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AN ADAPTIVE LOGIC BASED ON JAŚKOWSKI'S LOGIC D_2

MAREK NASIENIEWSKI*

1. *Introduction*

1.1. *Preliminaries*

In the present paper we construct an adaptive logic built over the logic D_2. The whole project of adaptive logics has been developed by Diderik Batens (see for example [3], [4], [6], and [8]). The idea is to use two consequence relations: a weaker one (called the *lower limit logic* and a stronger one (the *upper limit logic*). First two adaptive logics — today called ACLuN1 and ACLuN2 — were designed with CLuN (which is the full positive classical logic plus the law of excluded middle) as the lower limit logic[1] and classical logic (we use 'CL' as an abbreviation) as the upper limit logic. We recall here the logic CLuN not only for historical reasons. Some results[2] concerning the logic CLuN are useful also for stronger logics such as the logic D_2. Without going into details, let us only mention that in the case of the syntactics of adaptive logics, one can always use inferences valid for the lower limit logic but the inferences of the upper limit logic can only be applied under certain conditions. From a semantical point of view, the consequence relation of adaptive logics is richer than the consequence relation of the lower limit logic since only some CLuN-models of the set of premises are being selected. A crucial role is played by minimally inconsistent CLuN-consequences of the set of premises having the form $(A_1 \wedge \sim A_1) \vee \cdots \vee (A_n \wedge \sim A_n)$.

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[1] For the sake of simplicity we consider here only the propositional part of CLuN. Originally CLuN is meant as a first-order logic. Its propositional part appeared in [2] under the name 'PI'. The propositional adaptive logics were presented for the first time in [3].

[2] See Theorem 1.16, Lemma 1.17, and Corollary 1.18.

That a formula of this form is *minimally* inconsistent means here that no formula of the form $(A_{i_1} \wedge \sim A_{i_1}) \vee \cdots \vee (A_{i_m} \wedge \sim A_{i_m})$, where $1 \leq m < n$, is a CLuN-consequence of the set of premises. In the case of the logic ACLuN1 only CLuN-models of a given set of premises are allowed which validate only inconsistencies of the form $(B \wedge \sim B)$ which appear as disjuncts of some minimally inconsistent CLuN-consequences. In the case of the logic ACLuN2 only CLuN-models of a given set of premises are allowed, for which there is no CLuN-model of the given set of premises which, in a set-theoretical sense, validates fewer inconsistencies.[3]

An adaptive logic built in Jaśkowski's way has been already developed by Joke Meheus [18]. In the present paper we will use the same idea. However, we use the discussive interpretation of the connectives.

The main feature of the present adaptive logic is that it is formulated with the help of a modal language. That is why we use some basic notions and results from the field of modal logic[4]. To express this adaptive logic we use two consequence relations: the weaker one based on the logic S5 and the stronger one based on the logic Triv. The use of the modal language is connected with the fact that D_2, which is the lower limit logic of the adaptive logic under consideration, was originally expressed with the help of the modal logic S5 understood as a logic in the standard sense — i.e. as a certain set of formulas[5]. We consider here the consequence relation for the logic D_2. We recall some results[6] showing that while working with the so-called M-counterpart of the logic S5 it is enough to use some weaker modal logic called $\mathsf{S5}_\mathsf{M}$[7]. Moreover Definition 1.8, case 1 together with Corollary 1.10 prove that in order to work with our consequence relation, which is defined with the help of S5, it is enough to work with the above mentioned weaker modal logic. Similar remarks concern the M-counterpart of the logic Triv[8].

[3] For the exhaustive presentation of ACLuN1 and ACLuN2 see for example [6] or [8].

[4] Some standard notions such as a normal logic, a frame, the logic S5, the logic Triv, S5-frame, a valuation, truth in a given world, truth in a model, truth in a frame etc. can be found in the Appendix of [19].

[5] See [13].

[6] See Theorems 1.6 and 1.7 case 1.

[7] See Definition 1.5.

[8] See Definition 1.8 case 2, Theorem 1.11, Lemma 1.14, and Corollary 1.15.

Using Theorem 1.19 we can express meta-rules of inference for the adaptive logic based on D_2. As it was said, for this purpose we will use the M-counterpart of $\mathsf{S5}$. We apply Perzanowski's idea of treating inconsistencies as contingent formulas[9]. Let us consider a situation in which we have among our premises some inconsistent formulas. Of course, using classical logic we obtain the set of all formulas as a consequence of such an inconsistent set. To avoid this effect we can treat classical formulas as modal formulas using Jaśkowski's transformation: before each premise we will put '$\Diamond$', while each of the functors $\rightarrow$, $\wedge$, and $\leftrightarrow$ will be treated as a discussive one[10]. In our considerations transformations valid for $\mathsf{S5}$ are generally allowed. The procedure sketched above uses a kind of natural deduction for modal logics. Of course, at the end we will have to come back to the language without modalities. We will use — which is a standard property of adaptive logics — some conditional rules. Only in a limited way we can use transformations valid for the logic Triv. If, on the basis of the premises, some minimally contingent formula of the form $(\Diamond A_1 \wedge \Diamond \sim A_1) \vee \cdots \vee (\Diamond A_n \wedge \Diamond \sim A_n)$ (where $A_1 \ldots A_n$ are propositional variables[11]) is $\mathsf{S5}$-provable, then in the proof one cannot use Triv transformations, which apply non-trivially the assumption $\Diamond A_i \rightarrow \Box A_i$ for any $1 \leq i \leq n$.

A model of the logic CLuN has to fulfill — as far as the negation is concerned — the principle which says that for any two formulas, such that one is the negation of the other, at least one of them is true. Therefore, there are CLuN-models in which the negation of each CL-tautology is true. Thus, if among the premises there is a formula being the negation of some classical tautology then using an adaptive logic, which of course is not weaker than the lower limit logic, saves all premises as consequences and will not eliminate this feature of the logic CLuN. On the other hand in the case of the logic D_2, at least in some cases, the situation in which the negation of a CL-tautology is true, is not possible. If, for example, among the premises there is the negation of the law of excluded middle, then deductions from such premises lead to 'explosion', exactly as it is in the case of classical logic. It seems that this feature of D_2 is acceptable: if someone treats the negation of the law of excluded middle as a premise, then he should obtain triviality. Although it

[9] See [22].

[10] See Definition 1.9.

[11] Following J. Meheus, we have to impose a restriction on minimally contingent consequences since for example $\{A, \sim A, B\} \vdash_{\mathsf{D}_2} (B \wedge \sim B) \vee \big((\sim A \vee \sim B) \wedge \sim(\sim A \vee \sim B)\big)$, but this means that any assumption in the presence of inconsistent premises forms an inconsistent disjunct '$(B \wedge \sim B)$' of a minimally contingent consequence. We will use Meheus idea to limit ourselves to primitive formulas.

is possible to indicate examples of inconsistent formulas from which follow, on the basis of Jaśkowski's logic, other inconsistent formulas[12], one can talk about minimally inconsistent formulas being a consequence of a given set of premises.

1.2. *Some basic notions and results*

Let us recall the notion of $\mathsf{S5}$-consequence and $\mathsf{S5}$-provability:

Definition 1.1: $X \models_{\mathsf{S5}} A$ iff for each $\mathsf{S5}$-model $\langle W, R, v\rangle$: if $w \models_v X$, then $w \models_v A$.

Definition 1.2: $X \vdash_{\mathsf{S5}} A$ iff there is a finite sequence of formulas $C_1, \ldots, C_n$, where $C_n = A$ and for every $1 \leq i \leq n$ either $C_i \in X$, or C_i is a theorem of the logic $\mathsf{S5}$, or arises by the Modus Ponens rule (MP).

We have standard:

Theorem 1.3: *$X \models_{\mathsf{S5}} A$ iff $X \vdash_{\mathsf{S5}} A$.*

We will use an analogous definition for Triv-provability:

Definition 1.4: $X \vdash_{\mathsf{Triv}} A$ iff there is a finite sequence of formulas $C_1, \ldots,$ C_n where $C_n = A$ and for each $1 \leq i \leq n$ either $C_i \in X$, or C_i is a theorem of the logic Triv, or arises by Gödel's rule (GR)[13] or (MP).

We will use the following modal logic:

Definition 1.5: (Perzanowski, [21]) $\mathsf{S5_M}$ is the smallest normal logic containing:

$$\Box A \rightarrow \Diamond A \tag{D}$$

$$\Diamond\Box(\Diamond\Box A \rightarrow \Box A) \tag{M5}$$

$$\Diamond\Box(\Box A \rightarrow A) \tag{MT}$$

[12] The same is true for the logic CLuN: $(p \wedge \sim p) \wedge \sim(p \wedge \sim p) \vdash_{\mathsf{CLuN}} p \wedge \sim p$.

[13] Of course the rule (GR) can be skipped here.

closed under the following rule

$$\frac{\Diamond\Diamond A}{\Diamond A} \tag{RT*}$$

We will need the following axioms:

$$\Box\Diamond\Diamond A \to \Diamond A \tag{T*}$$

$$\Box\Diamond A \to \Diamond A \tag{D*}$$

$$\Box\Diamond A \to \Diamond\Box A^{14} \tag{K1}$$

For any modal logic *P*, and a set of modal formulas X, by $P[X]$ we mean the smallest normal logic containing the logic *P* and including all formulas from X. If $X = \{A_1, \dots, A_n\}$ we will write $P[A_1 \dots A_n]$.

Let $\mathrm{For}^{\mathrm{M}}$ denote the set of all modal propositional formulas and For denote the set of all classical propositional formulas. We have:

Theorem 1.6: (Dziobiak, [9]) $\mathsf{S5}_{\mathsf{M}} = \mathsf{K}[\mathrm{D}^*\mathrm{T}^*]$

Let $\mathrm{M}(P) := \{A \in \mathrm{For}^{\mathrm{M}} : \Diamond A \in P\}$. The set of formulas $\mathrm{M}(P)$ we call the M-counterpart of the logic *P*.

In [21] it was shown that

Theorem 1.7: (Perzanowski, [21]) *(1)* $\mathrm{M}(\mathsf{S5}) = \mathrm{M}(\mathsf{S5}_{\mathsf{M}})$.
(2) $\mathrm{M}(\mathsf{S4}[\mathrm{K1}]) = \mathsf{Triv}$.

As an analogue to the notion of M-counterpart of the logic $\mathsf{S5}$ we can define the following consequence relations:

Definition 1.8: (1) $X \vdash_{\Diamond\mathsf{S5}} A$ iff $\{\Diamond B : B \in X\} \vdash_{\mathsf{S5}} \Diamond A$.
(2) $X \vdash_{\Diamond\mathsf{Triv}} A$ iff $\{\Diamond B : B \in X\} \vdash_{\mathsf{Triv}} \Diamond A$.

Let us recall

14 The McKinsey-Sobociński axiom is denoted in [15] as 'M'. In [23], Sobociński by *K1* denotes a formula $\Box(\Box\Diamond p \wedge \Box\Diamond q \to \Diamond(p \wedge q))$. He proves that this formula is equivalent on the basis of $\mathsf{S4}$, to the formula $\Box(\Box\Diamond p \to \Diamond\Box p)$ giving another axiomatisation of McKinsey's system $\mathsf{S4.1}$, which he himself calls K1.

Definition 1.9: By *Jaśkowski's transformation* we mean the function $-^{\mathrm{d}}\colon \mathrm{For} \longrightarrow \mathrm{For}^{\mathrm{M}}$ from the set of all propositional formulas into the set of all modal propositional formulas, defined by induction for any $A \in \mathrm{For}$:

(1) If A is a propositional variable, then $A^{\mathrm{d}} = A$
(2) (a) if A is of the form $B \vee C$, then $A^{\mathrm{d}} = B^{\mathrm{d}} \vee C^{\mathrm{d}}$
 (b) if A is of the form $B \wedge C$, then $A^{\mathrm{d}} = B^{\mathrm{d}} \wedge \Diamond C^{\mathrm{d}}$
 (c) if A is of the form $B \rightarrow C$, then $A^{\mathrm{d}} = \Diamond B^{\mathrm{d}} \rightarrow C^{\mathrm{d}}$
 (d) if A is of the form $B \leftrightarrow C$, then $A^{\mathrm{d}} = (\Diamond B^{\mathrm{d}} \rightarrow C^{\mathrm{d}}) \wedge \Diamond(\Diamond C^{\mathrm{d}} \rightarrow B^{\mathrm{d}})$
 (e) if A is of the form $\sim B$, then $A^{\mathrm{d}} = \sim(B^{\mathrm{d}})$.

It is easy to see that:

Corollary 1.10: $\vdash_{\mathsf{S5_M}} \Diamond(\ldots((A_1 \wedge A_2) \wedge A_3) \wedge \cdots \wedge A_n) \rightarrow A)^{\mathrm{d}}$ *iff* $\{A_1^{\mathrm{d}}, \ldots, A_n^{\mathrm{d}}\} \vdash_{\Diamond\mathsf{S5}} A^{\mathrm{d}}$.

Proof. Assume that $\vdash_{\mathsf{S5_M}} \Diamond(\ldots((A_1 \wedge A_2) \wedge A_3) \wedge \cdots \wedge A_n) \rightarrow A)^{\mathrm{d}}$. By Theorem 1.7 it is equivalent to the fact that $\vdash_{\mathsf{S5}} \Diamond(\ldots((A_1 \wedge A_2) \wedge A_3) \wedge \cdots \wedge A_n) \rightarrow A)^{\mathrm{d}}$. By the definition of the operation $-^{\mathrm{d}}$ using the laws of S5 we see that the last statement holds iff $\vdash_{\mathsf{S5}} (\ldots((\Diamond A_1^{\mathrm{d}} \wedge \Diamond A_2^{\mathrm{d}}) \wedge \Diamond A_3^{\mathrm{d}}) \wedge \cdots \wedge \Diamond A_n^{\mathrm{d}}) \rightarrow \Diamond A^{\mathrm{d}}$ but by Definitions 1.2 and 1.8, and the Deduction Theorem this holds iff $\{A_1^{\mathrm{d}}, \ldots, A_n^{\mathrm{d}}\} \vdash_{\Diamond\mathsf{S5}} A^{\mathrm{d}}$. $\Box$

The above remark shows that our considerations presented in section 2 can be expressed with the help of logic $\mathsf{S5_M}$.

We also have:

Theorem 1.11: *The logic* $\mathsf{S5_M}[\mathrm{K1}]$ *is the smallest normal modal logic P, for which* $\mathsf{Triv} = \mathrm{M}(P)$.

Proof. Consider a normal modal logic *P*, such that $\mathsf{Triv} = \mathrm{M}(P)$. We easily see that $\mathsf{S5_M}[\mathrm{K1}] \subseteq P$. Indeed, let us notice that (T^*), (D^*) and $(\mathrm{K1})$ are theorems of *P*. On the basis of the logic K these formulas are equivalent respectively to: $\Diamond(\Diamond\Diamond A \rightarrow A)$, $\Diamond(\Diamond A \rightarrow A)$ and $\Diamond(\Diamond A \rightarrow \Box A)$. Of course $\Diamond\Diamond A \rightarrow A$, $\Diamond A \rightarrow A$, $\Diamond A \rightarrow \Box A \in \mathsf{Triv}$. Thus, since $\mathsf{Triv} = \mathrm{M}(P)$, we have that (T^*), (D^*) and $(\mathrm{K1})$ are theorems of *P*, but this means that $\mathsf{S5_M}[\mathrm{K1}] \subseteq P$.

Now we prove that $\mathsf{Triv} = \mathrm{M}(\mathsf{S5_M}[\mathrm{K1}])$. In this way the postulated minimality will be proved.

First, we show that $\mathsf{Triv} \subseteq \mathrm{M}(\mathsf{S5_M}[\mathrm{K1}])$. Let $A \in \mathsf{Triv}$. Assume that $\{\Diamond A_i\}_{1 \leq i \leq n} \cup \{\Box B_j\}_{1 \leq j \leq m}$ is the set of all subformulas of the formula A, whose main functors are respectively '$\Diamond$' and '$\Box$'. Let A' be a formula arising by elimination of all modal functors from A. By a standard characterization of the logic Triv we have that $A' \in \mathsf{CL}$. It is easy to see that $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Box B \to \Diamond\Box B$ by (D^*). On the other hand *via* the monotonicity rule and the axiom (D): $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Box B \to \Diamond B$ we see that $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond\Box B \to \Diamond\Diamond B$, thus by transitivity of implication we have: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Box B \to \Diamond\Diamond B$. The last statement is equivalent to: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond(B \to \Diamond B)$. While by the axiom (D^*): $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond(\Diamond B \to B)$, with the help of the following rule

$$(\sharp) \qquad \frac{\Diamond B_1, \ldots, \Diamond B_k}{\Diamond(B_1 \wedge \cdots \wedge B_k)}$$

which is admissible for $\mathsf{S5_M}[\mathrm{K1}]$[15] we have: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond(B \leftrightarrow \Diamond B)$. Analogously we prove that $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond(B \leftrightarrow \Box B)$. Applying the rule $(\sharp)$ for formulas $\Diamond(A_i \leftrightarrow \Diamond A_i)$, $1 \leq i \leq n$; $\Diamond(B_j \leftrightarrow \Box B_j)$, $1 \leq j \leq m$ and $\Diamond A'$ we obtain $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond\big((A_1 \leftrightarrow \Diamond A_1) \wedge \cdots \wedge (A_n \leftrightarrow \Diamond A_n) \wedge (B_1 \leftrightarrow \Box B_1) \wedge \cdots \wedge (B_m \leftrightarrow \Box B_m) \wedge A'\big)$. By extensionality we deduce that: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond A$. This proves that $\mathsf{Triv} \subseteq \mathrm{M}(\mathsf{S5_M}[\mathrm{K1}])$.

In [20] it was shown that $\mathsf{S5_M}[\mathrm{K1}] \subsetneq \mathsf{K}4\mathrm{M}$[16]. Of course $\mathsf{K}4\mathrm{M} \subseteq \mathsf{S4}[\mathrm{K1}]$. Therefore $\mathrm{M}(\mathsf{S5_M}[\mathrm{K1}]) \subseteq \mathrm{M}(\mathsf{S4}[\mathrm{K1}])$ and by Theorem 1.7 we have that $\mathrm{M}(\mathsf{S4}[\mathrm{K1}]) = \mathsf{Triv}$. Thus, $\mathrm{M}(\mathsf{S5_M}[\mathrm{K1}]) \subseteq \mathsf{Triv}$. □

Let for any set X of propositional formulas $\mathsf{J}(X) = \mathsf{K}[\{\Diamond A^{\mathrm{d}} : A \in X\}]$. We have

Theorem 1.12: ([19]) $\mathsf{S5_M} = \mathsf{J}(\mathsf{CLuN})$.

Corollary 1.13: ([20]) $\mathsf{S5_M}[\mathrm{K1}] = \mathsf{J}(\mathsf{CL})$.

Corollary 1.13 expresses the fact that logic $\mathsf{S5_M}[\mathrm{K1}]$ is the minimal normal modal logic which defines the classical logic in the discussive way:

Lemma 1.14: *The set $\mathcal{K}$ of all classical propositional formulas, arising from modal formulas A by replacement in A of all appearances of subformulas of the form $\Diamond C \to D$, $C \wedge \Diamond D$, and $\big(\Diamond C \to D\big) \wedge \Diamond\big(\Diamond D \to C\big)$ by formulas of the form $C \to D$, $C \wedge D$, and $C \leftrightarrow D$ respectively, such that $\Diamond A \in$*

[15] This rule follows from Theorem $(\star)$: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Box(\Box\Diamond B \wedge \Box\Diamond C) \to \Diamond(B \wedge C)$ proved in the proof of the theorem 6 in [20].

[16] $\mathsf{K}4\mathrm{M}$ is the smallest normal logic containing axiom (4) and (K1).

$\mathsf{S5_M}$[K1], *forms the classical logic, i.e. for any* $B \in$ For*:* $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond B^{\mathrm{d}}$ *iff* $\vdash_{\mathsf{CL}} B$.

Proof. By Corollary 1.13 we see that $\mathcal{K}$ includes every theorem of CL.

If a given $B \in$ For is not a theorem of CL, then there is no modal formula arising from B by addition of some '$\Diamond$', which would be a theorem of Triv, so by Theorem 1.11 in particular $B^{\mathrm{d}} \notin \mathsf{Triv}$ i.e. $\Diamond B^{\mathrm{d}} \notin \mathsf{S5_M}[\mathrm{K1}]$. □

The following corollary is an analogue of Corollary 1.10. We omit the obvious proof.

Corollary 1.15: $\vdash_{\mathsf{S5_M}[\mathsf{K1}]} \Diamond(\ldots((A_1 \wedge A_2) \wedge A_3) \wedge \cdots \wedge A_n) \to A)^{\mathrm{d}}$ *iff* $\{A_1^{\mathrm{d}}, \ldots, A_n^{\mathrm{d}}\} \vdash_{\Diamond\mathsf{Triv}} A^{\mathrm{d}}$.

Let us recall

Theorem 1.16: (Batens, [3]) *If* $\vdash_{\mathsf{CL}} A$*, then* $\vdash_{\mathsf{CLuN}} (C_1 \wedge {\sim} C_1) \vee \cdots \vee (C_n \wedge {\sim} C_n) \vee A$*, where* $\{{\sim} C_1, \ldots, {\sim} C_n\}$ *is the set of all negative subformulas of formula* A*.*

This theorem easily entails the following:

Lemma 1.17: *Let L be any propositional logic such that* $\mathsf{CLuN} \subseteq L \subseteq \mathsf{CL}$*. We have:* $\vdash_{\mathsf{CL}} A$ *iff* $\vdash_L (A_1 \wedge {\sim} A_1) \vee \cdots \vee (A_n \wedge {\sim} A_n) \vee A$*, where* ${\sim} A_1$*, …,* ${\sim} A_n$ *(*$n \geq 0$*) are all negative subformulas of the formula* A*.*

Proof. ($\Rightarrow$). We assume that $\vdash_{\mathsf{CL}} A$. By Theorem 1.16 we have $\vdash_{\mathsf{CLuN}} (A_1 \wedge {\sim} A_1) \vee \cdots \vee (A_n \wedge {\sim} A_n) \vee A$. By the assumption we have $\mathsf{CLuN} \subseteq L$, therefore $\vdash_L (A_1 \wedge {\sim} A_1) \vee \cdots \vee (A_n \wedge {\sim} A_n) \vee A$.

($\Leftarrow$). By the assumption, each theorem of the logic L is a theorem of classical logic, and therefore, since $\vdash_L (A_1 \wedge {\sim} A_1) \vee \cdots \vee (A_n \wedge {\sim} A_n) \vee A$, so also $\vdash_{\mathsf{CL}} (A_1 \wedge {\sim} A_1) \vee \cdots \vee (A_n \wedge {\sim} A_n) \vee A$. But for any classical valuation inconsistent formulas are false, thus $\vdash_{\mathsf{CL}} A$. □

By the above lemma, since $\mathsf{CLuN} \subseteq \mathsf{D_2} \subseteq \mathsf{CL}$, we have the following:

Corollary 1.18: $\vdash_{\mathsf{CL}} A$ *iff* $\vdash_{\mathsf{D_2}} (A_1 \wedge {\sim} A_1) \vee \cdots \vee (A_n \wedge {\sim} A_n) \vee A$*, where* $\{{\sim} A_i\}_{1 \leq i \leq n}$ *is the set of all negative subformulas of the formula* A*.*

Theorem 1.19: *Let* $A \in For$ *and* $\{A_i\}_{1 \leq i \leq n}$ *be the set of all subformulas of the formula* A*, having a form* ${\sim} B_i$ *for some formula* B_i*. Then:*
$\vdash_{\mathsf{S5}} \big((\Diamond A_1^{\mathrm{d}} \to \Box A_1^{\mathrm{d}}) \wedge \cdots \wedge (\Diamond A_n^{\mathrm{d}} \to \Box A_n^{\mathrm{d}})\big) \to \Diamond A^{\mathrm{d}}$ *iff* $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond A^{\mathrm{d}}$.

Proof. ($\Rightarrow$) We assume that $\vdash_{\mathsf{S5}} \big((\Diamond A_1^{\mathrm{d}} \to \Box A_1^{\mathrm{d}}) \wedge \cdots \wedge (\Diamond A_n^{\mathrm{d}} \to \Box A_n^{\mathrm{d}})\big) \to \Diamond A^{\mathrm{d}}$. Using the fact that for any $1 \leq i \leq n$: $\vdash_{\mathsf{S5}} (\Diamond A_i^{\mathrm{d}} \to \Box A_i^{\mathrm{d}}) \to (\Diamond A_i^{\mathrm{d}} \to \Box A_i^{\mathrm{d}})$, and the usual reductions of modalities for $\mathsf{S5}$ we have: $\vdash_{\mathsf{S5}} \Box\Diamond(\Diamond A_i^{\mathrm{d}} \to \Box A_i^{\mathrm{d}}) \to (\Diamond A_i^{\mathrm{d}} \to \Box A_i^{\mathrm{d}})$; by multiplying these implications we obtain: $\vdash_{\mathsf{S5}} \Box\big(\Diamond(\Diamond A_1^{\mathrm{d}} \to \Box A_1^{\mathrm{d}}) \wedge \cdots \wedge \Diamond(\Diamond A_n^{\mathrm{d}} \to \Box A_n^{\mathrm{d}})\big) \to \big((\Diamond A_1^{\mathrm{d}} \to \Box A_1^{\mathrm{d}}) \wedge \cdots \wedge (\Diamond A_n^{\mathrm{d}} \to \Box A_n^{\mathrm{d}})\big)$, while by the transitivity of the implication using the assumption we see that: $\vdash_{\mathsf{S5}} \Box\big(\Diamond(\Diamond A_1^{\mathrm{d}} \to \Box A_1^{\mathrm{d}}) \wedge \cdots \wedge \Diamond(\Diamond A_n^{\mathrm{d}} \to \Box A_n^{\mathrm{d}})\big) \to \Diamond A^{\mathrm{d}}$, so by Theorem 1.7: $\vdash_{\mathsf{S5}_\mathsf{M}} \Box\big(\Diamond(\Diamond A_1^{\mathrm{d}} \to \Box A_1^{\mathrm{d}}) \wedge \cdots \wedge \Diamond(\Diamond A_n^{\mathrm{d}} \to \Box A_n^{\mathrm{d}})\big) \to \Diamond A^{\mathrm{d}}$. Obviously we have: $\vdash_{\mathsf{S5}_\mathsf{M}[\mathrm{K1}]} \Diamond(\Diamond A_i^{\mathrm{d}} \to \Box A_i^{\mathrm{d}})$, thus also $\vdash_{\mathsf{S5}_\mathsf{M}[\mathrm{K1}]} \Diamond(\Diamond A_1^{\mathrm{d}} \to \Box A_1^{\mathrm{d}}) \wedge \cdots \wedge \Diamond(\Diamond A_n^{\mathrm{d}} \to \Box A_n^{\mathrm{d}})$. Therefore by the rule (GR) and (MP) we receive $\vdash_{\mathsf{S5}_\mathsf{M}[\mathrm{K1}]} \Diamond A^{\mathrm{d}}$.

($\Leftarrow$). Let's assume that $\vdash_{\mathsf{S5}_\mathsf{M}[\mathrm{K1}]} \Diamond A^{\mathrm{d}}$. By Lemma 1.14 we have that A is a theorem of CL. Taking into account Lemma 1.18 let us consider the set $\{A_i\}_{1 \leq i \leq n}$ of all negative subformulas of the formula A. We see that $\vdash_{\mathsf{D}_2} (A_1 \wedge \sim A_1) \vee \cdots \vee (A_n \wedge \sim A_n) \vee A$. On the basis of the definition of the logic D_2 it means that $\vdash_{\mathsf{S5}} \Diamond\big((A_1^{\mathrm{d}} \wedge \Diamond \sim A_1^{\mathrm{d}}) \vee \cdots \vee (A_n^{\mathrm{d}} \wedge \Diamond \sim A_n^{\mathrm{d}}) \vee A^{\mathrm{d}}\big)$. By the laws of distributivity of '$\Diamond$' with respect to '$\vee$' and '$\wedge$' we have: $\vdash_{\mathsf{S5}} \big((\Diamond A_1^{\mathrm{d}} \wedge \Diamond \sim A_1^{\mathrm{d}}) \vee \cdots \vee (\Diamond A_n^{\mathrm{d}} \wedge \Diamond \sim A_n^{\mathrm{d}})\big) \vee \Diamond A^{\mathrm{d}}$, and by de Morgan's law using CL we have: $\vdash_{\mathsf{S5}} \big((\Diamond A_1^{\mathrm{d}} \to \Box A_1^{\mathrm{d}}) \wedge \cdots \wedge (\Diamond A_n^{\mathrm{d}} \to \Box A_n^{\mathrm{d}})\big) \to \Diamond A^{\mathrm{d}}$. $\Box$

2. *A formulation of an adaptive logic over* D_2

We introduce in a formal way an adaptive logic on the basis of Jaśkowski's logic. We will use the following definition by Joke Meheus [18]:

Definition 2.1: Let $\{A_i\}_{1 \leq i \leq n}$ be a set of propositional variables. A formula $(\Diamond A_1 \wedge \Diamond \sim A_1) \vee \cdots \vee (\Diamond A_n \wedge \Diamond \sim A_n)$ $\mathsf{S5}$-provable on the basis of the set of premises X is called *a minimally contingent* $\mathsf{S5}$*-consequence of the set* X[17] iff no formula of the form $(\Diamond A_{i_1} \wedge \Diamond \sim A_{i_1}) \vee \cdots \vee (\Diamond A_{i_m} \wedge \Diamond \sim A_{i_m})$, for $1 \leq i_1, \ldots, i_m \leq n$ where $m < n$, is $\mathsf{S5}$-provable on the basis of the set X.

We prove an obvious, but useful lemma:

[17] Analogously we can define *a minimally contingent semantical* $\mathsf{S5}$*-consequence of a given set* X. Of course both notions are equivalent.

Lemma 2.2: *Let X be an* $\mathsf{S5}$*-consistent set of formulas. For any $n \geq 1$ and any propositional variables B_i, where $1 \leq i \leq n$, if the formula $(\Diamond B_1 \wedge \Diamond \sim B_1) \vee \cdots \vee (\Diamond B_n \wedge \Diamond \sim B_n)$ is a minimally contingent* $\mathsf{S5}$*-consequence of the set X, then for each $1 \leq i \leq n$ a variable B_i is a subformula of some member of the set X.*

Proof. We assume that a formula $(\Diamond B_1 \wedge \Diamond \sim B_1) \vee \cdots \vee (\Diamond B_n \wedge \Diamond \sim B_n)$ is a minimally contingent $\mathsf{S5}$-consequence of the set X. We also assume for contradiction that, for example, a propositional variable B_1 does not appear in any of formulas from the set X[18]. Using the completeness theorem and the minimality condition we see that $X \not\models_{\mathsf{S5}} (\Diamond B_2 \wedge \Diamond \sim B_2) \vee \cdots \vee (\Diamond B_n \wedge \Diamond \sim B_n)$. Thus, there is an $\mathsf{S5}$-model $\langle W, R, v\rangle$ and a world $w \in W$ such the set X is true in w, while a formula $(\Diamond B_2 \wedge \Diamond \sim B_2) \vee \cdots \vee (\Diamond B_n \wedge \Diamond \sim B_n)$ is false in w. We define a model $\langle W, R, v'\rangle$, where $v'(B_1) = W$ and for any other propositional variable C: $v'(C) = v(C)$. By the standard inductive argument we can show that for any formula B in which B_1 does not appear and for any $w_1 \in W$ we have: $w_1 \models_v B$ iff $w_1 \models_{v'} B$. Thus $w \models_{v'} X$ and $w \not\models_{v'} (\Diamond B_2 \wedge \Diamond \sim B_2) \vee \cdots \vee (\Diamond B_n \wedge \Diamond \sim B_n)$. We also have that for any $w_1 \in W$: $w_1 \not\models_{v'} (\Diamond B_1 \wedge \Diamond \sim B_1)$. But in particular for the world w it means that $w \not\models_{v'} (\Diamond B_1 \wedge \Diamond \sim B_1) \vee \cdots \vee (\Diamond B_n \wedge \Diamond \sim B_n)$. But this means that $X \not\models_{\mathsf{S5}} (\Diamond B_1 \wedge \Diamond \sim B_1) \vee \cdots \vee (\Diamond B_n \wedge \Diamond \sim B_n)$. By the completeness theorem we obtain a contradiction. $\square$

Definition 2.3: $X \vdash_{\mathsf{D_2}} A$ iff $X^{\mathrm{d}} = \{B^{\mathrm{d}} : B \in X\} \vdash_{\Diamond\mathsf{S5}} A^{\mathrm{d}}$.

Definition 2.4: We say that *a formula A is* $\mathsf{AD_2}$*-provable on the basis of the set X* (notation $X \vdash_{\mathsf{AD_2}} A$) iff either

(1) $A \in X$ or[19]
(2) $X \vdash_{\mathsf{D_2}} A$ or
(3) there is a CL-proof of the formula A on the basis of X, where for $n \geq 1, m \geq 1$, some set $\{A_1, \ldots, A_n\} \subseteq X$, and a set $\{B_1, \ldots, B_m\}$ of propositional variables, the following holds: $\vdash_{\mathsf{D_2}} \big((B_1 \wedge \sim B_1) \vee \cdots \vee (B_m \wedge \sim B_m)\big) \vee \big((A_1 \wedge \cdots \wedge A_n) \rightarrow A\big)$ and for each $1 \leq i \leq m$ no formula of the form $\Diamond B_i \wedge \Diamond \sim B_i$ is a disjunct of any minimally contingent $\mathsf{S5}$-consequence of $\Diamond X^{\mathrm{d}}$.

The above formulation is an analogue of the logic $\mathsf{ACLuN1}$ and uses an equivalent formulation of its syntactic consequence relation.

[18] The proof for other B_i is exactly the same.

[19] Of course, this case can be skipped.

Using Definition 1.1 of semantical S5-consequence we can introduce the following notion:

Definition 2.5: An S5-model $M = \langle W, R, v\rangle$ is an AD_2-*model with respect to the set of premises* X iff for any world $w \in W$ and for any propositional variable A if $w \models_v \Diamond(A \wedge \sim A)^{\mathrm{d}}$, then there are $n \geq 0$ and propositional variables $A_1, \ldots, A_n$, such that $\Diamond\big((A \wedge \sim A) \vee (A_1 \wedge \sim A_1) \vee \cdots \vee (A_n \wedge \sim A_n)\big)^{\mathrm{d}}$ is a minimally contingent semantical S5-consequence of the set $\Diamond X^{\mathrm{d}}$.

Definition 2.6: We say that a formula A is an AD_2-*consequence of the set* X iff for any AD_2-model $M = \langle W, R, v\rangle$ with respect to the set X and any $w \in W$, if all formulas of the set $\Diamond X^{\mathrm{d}}$ are true in w, then $\Diamond A^{\mathrm{d}}$ is true in w, (notation $X \models_{\mathsf{AD}_2} A$).

Theorem 2.7: (Soundness of AD_2) *If* $X \vdash_{\mathsf{AD}_2} A$*, then* $X \models_{\mathsf{AD}_2} A$.

Proof. Assume that $X \vdash_{\mathsf{AD}_2} A$. We show that for any AD_2-model M with respect to the set X, and any world w from the set of possible worlds of M: if all formulas of the set $\Diamond X^{\mathrm{d}}$ are true in w, then the formula $\Diamond A^{\mathrm{d}}$ is also true in w. Assume that all formulas of the set $\Diamond X^{\mathrm{d}}$ are true in a given world w. We consider the following cases:

(1) $A \in X$, then obviously $w \models_v \Diamond A^{\mathrm{d}}$.

(2) $\Diamond A^{\mathrm{d}}$ is S5-provable on the basis of the set $\Diamond X^{\mathrm{d}}$; since the given model is in particular an S5-model of the set $\Diamond X^{\mathrm{d}}$, thus by Theorem 1.3 we have $w \models_v \Diamond A^{\mathrm{d}}$.

(3) there is a CL-proof of a formula A on the basis of the set X, such that there are propositional variables B_i, $1 \leq i \leq m$ and $C_j \in X$, $1 \leq j \leq k$ for which $\vdash_{\mathsf{D}_2} \big((B_1 \wedge \sim B_1) \vee \cdots \vee (B_m \wedge \sim B_m)\big) \vee (C_1 \wedge \cdots \wedge C_k \rightarrow A)$, where for $1 \leq i \leq m$ none of $\Diamond(B_1 \wedge \sim B_i)^{\mathrm{d}}$ is a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}$. By the definition of the logic D_2 we have that $\vdash_{\mathsf{S5}} \Diamond\big[\big((B_1 \wedge \Diamond \sim B_1) \vee \cdots \vee (B_m \wedge \Diamond \sim B_m)\big) \vee \big(\Diamond(C_1^{\mathrm{d}} \wedge \Diamond C_1^{\mathrm{d}} \wedge \cdots \wedge \Diamond C_k^{\mathrm{d}}) \rightarrow A^{\mathrm{d}}\big)\big]$, we have also $\vdash_{\mathsf{S5}} \big((\Diamond B_1 \wedge \Diamond \sim B_1) \vee \cdots \vee (\Diamond B_m \wedge \Diamond \sim B_m)\big) \vee (\Diamond C_1^{\mathrm{d}} \wedge \cdots \wedge \Diamond C_k^{\mathrm{d}} \rightarrow \Diamond A^{\mathrm{d}})$. By the definition 2.5 of an AD_2-model none of the formulas $\Diamond B_i \wedge \Diamond \sim B_i$ is true in the world w — if it were, $\Diamond B_i \wedge \Diamond \sim B_i$ would be a disjunct of a minimally contingent semantical S5-consequence of the set $\Diamond X^{\mathrm{d}}$, contrary to the choice of B_i, by Theorem 1.3. It means that the formula $(\Diamond C_1^{\mathrm{d}} \wedge \cdots \wedge \Diamond C_k^{\mathrm{d}} \rightarrow \Diamond A^{\mathrm{d}})$ is true in the world w. Since all formulas of the set $\Diamond X^{\mathrm{d}}$ are true in the world

w, therefore in particular all formulas $\Diamond C_1^{\mathrm{d}}$, …, $\Diamond C_k^{\mathrm{d}}$ are true in w. Thus $\Diamond A^{\mathrm{d}}$ is true in w as well.

□

Theorem 2.8: (Completeness theorem) *If* $X \models_{\mathsf{AD_2}} A$*, then* $X \vdash_{\mathsf{AD_2}} A$.

Proof. Assume that $X \nvdash_{\mathsf{AD_2}} A$. By Definition 2.4 neither of its three cases holds. In particular this means that also $X \nvdash_{\mathsf{D_2}} A$. But by Definition 2.3 and the completeness theorem for S5 we obtain $\Diamond X^{\mathrm{d}} \not\models_{\mathsf{S5}} \Diamond A^{\mathrm{d}}$. So, there is an S5-model $M = \langle W, R, v\rangle$ and a world $w \in W$ such that $w \models_v \Diamond X^{\mathrm{d}}$ while $w \not\models_v \Diamond A^{\mathrm{d}}$. We consider a set $\{\sim(\Diamond B \wedge \Diamond \sim B) :$ B is a propositional variable and $(\Diamond B \wedge \Diamond \sim B)$ is not a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}\}$. If this set is empty, then for any propositional variable B, a formula $(\Diamond B \wedge \Diamond \sim B)$ is a disjunct of some minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}$. But by the completeness theorem for S5 this means that for any S5-model, the condition of Definition 2.5 is fulfilled, thus M is an $\mathsf{AD_2}$-model, so $X \not\models_{\mathsf{AD_2}} A$.

Therefore we can assume that the set $\{\sim(\Diamond B \wedge \Diamond \sim B) :$ B is a propositional variable and $(\Diamond B \wedge \Diamond \sim B)$ is not a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}\}$ is not empty. We construct $\mathsf{AD_2}$-model M with respect to the set X, such that the set $\Diamond X^{\mathrm{d}}$ is validated in some world of M, in which the formula $\Diamond A^{\mathrm{d}}$ is falsified. Let's notice that

($\star$) The set $\Diamond X^{\mathrm{d}} \cup \{\sim(\Diamond B \wedge \Diamond \sim B) :$ B is a propositional variable and $(\Diamond B \wedge \Diamond \sim B)$ is not a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}\} \cup \{\sim \Diamond A^{\mathrm{d}}\}$ is S5-consistent.

Indeed, assume for contradiction that there are formulas $A_1, \ldots, A_n \in X$ and propositional variables $B_1, \ldots, B_m$, such that none of $(\Diamond B_i \wedge \Diamond \sim B_i)$ for $1 \leq i \leq m$ is a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}$ and $\vdash_{\mathsf{S5}} \sim\big(\Diamond A_1^{\mathrm{d}} \wedge \cdots \wedge \Diamond A_n^{\mathrm{d}} \wedge \sim(\Diamond B_1 \wedge \Diamond \sim B_1) \wedge \cdots \wedge \sim(\Diamond B_m \wedge \Diamond \sim B_m) \wedge \sim \Diamond A^{\mathrm{d}}\big)$. *Via* classical logic we have $\vdash_{\mathsf{S5}} \Diamond A_1^{\mathrm{d}} \wedge \cdots \wedge \Diamond A_n^{\mathrm{d}} \to (\Diamond B_1 \wedge \Diamond \sim B_1) \vee \cdots \vee (\Diamond B_m \wedge \Diamond \sim B_m) \vee \Diamond A^{\mathrm{d}}$ and $\vdash_{\mathsf{S5}} (\Diamond B_1 \wedge \Diamond \sim B_1) \vee \cdots \vee (\Diamond B_m \wedge \Diamond \sim B_m) \vee (\Diamond A_1^{\mathrm{d}} \wedge \cdots \wedge \Diamond A_n^{\mathrm{d}} \to \Diamond A^{\mathrm{d}})$, but by Definition 2.4 of the logic $\mathsf{AD_2}$ it would mean that $X \vdash_{\mathsf{AD_2}} A$, and this would contradict the assumption. By the Lindenbaum's lemma each consistent set with respect to a given logic is included in some maximally consistent set. Let us consider the set W of all maximally S5-consistent sets containing the set $\{\sim(\Diamond B \wedge \Diamond \sim B) :$ B is a propositional variable and

$(\Diamond B \wedge \Diamond \sim B)$ is not a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}\}$.

We define a model M, putting W as the set of possible worlds, the accessibility relation R is defined as the accessibility relation of the canonical frame i.e. for each $w, w' \in W$: wRw' iff for any $A \in \mathrm{For}^{\mathrm{M}}$ if $\Box A \in w$, then $A \in w'$.

Of course the accessibility relation R is reflexive, symmetric, and transitive (the proof is standard), so $\langle W, R\rangle$ is an S5-frame.

A valuation function is defined, as usual, as the valuation in a canonical model i.e. for any propositional variable B and any $w \in W$: $w \models_v B$ iff $B \in w$. By the standard inductive argument we prove that:

(†) for any formula C and any $w \in W$: $w \models_v C$ iff $C \in w$.

Let us mention only the following case: $w \models_v \Box C$ implies $\Box C \in w$. The other cases can be proved in the standard way. We assume that $w \models_v \Box C$ and $\Box C \notin w$. We prove that the set $\mathcal{W} = \{D : \Box D \in w\} \cup \{\sim C\} \cup \{\sim(\Diamond B \wedge \Diamond \sim B) : B$ is a propositional variable and $(\Diamond B \wedge \Diamond \sim B)$ is not a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}\}$ is S5-consistent. If it were not consistent we would have that $\vdash_{\mathsf{S5}} \sim\big(D_1 \wedge \cdots \wedge D_n \wedge \sim(\Diamond B_1 \wedge \Diamond \sim B_1) \wedge \cdots \wedge \sim(\Diamond B_m \wedge \Diamond \sim B_m) \wedge \sim C\big)$, for some $\Box D_1, \ldots, \Box D_n \in w$ and propositional variables $B_1, \ldots, B_m$, such that none of $(\Diamond B_i \wedge \Diamond \sim B_i)$ for $1 \leq i \leq m$ is a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}$. By the classical logic we have that $\vdash_{\mathsf{S5}} \big(D_1 \wedge \cdots \wedge D_n \wedge \sim(\Diamond B_1 \wedge \Diamond \sim B_1) \wedge \cdots \wedge \sim(\Diamond B_m \wedge \Diamond \sim B_m)\big) \rightarrow C$. By monotonicity and the usual reductions of modalities for S5 we have $\vdash_{\mathsf{S5}} \big(\Box D_1 \wedge \cdots \wedge \Box D_n \wedge \sim(\Diamond B_1 \wedge \Diamond \sim B_1) \wedge \cdots \wedge \sim(\Diamond B_m \wedge \Diamond \sim B_m)\big) \rightarrow \Box C$. But by the definition of W and the maximality of w, we have that $\Box D_1 \wedge \cdots \wedge \Box D_n \wedge \sim(\Diamond B_1 \wedge \Diamond \sim B_1) \wedge \cdots \wedge \sim(\Diamond B_m \wedge \Diamond \sim B_m) \in w$ and also $\Box C \in w$. We have a contradiction. Thus, $\mathcal{W}$ is S5-consistent and there is $w_1 \in W$ such, that $\mathcal{W} \subseteq w_1$ and wRw_1. Of course $w_1 \models_v C$, since $w \models_v \Box C$. By the induction hypothesis for w_1, we have that $C \in w_1$, but $\sim C \in \mathcal{W} \subseteq w_1$, which is a contradiction since w_1 is consistent.

We prove that the defined model is an AD_2-model with respect to the set of premises $\Diamond X^{\mathrm{d}}$. Let's assume that for some propositional variable B, the formula $\Diamond(B \wedge \Diamond \sim B)$ is true in some world w of the considered S5-model. If $\Diamond(B \wedge \Diamond \sim B)$ were not a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}$, then according to the construction of M the formula $\sim \Diamond(B \wedge \Diamond \sim B)$ would be true in the world w, which is impossible. Thus, for each world w and for every propositional variable B, the formula $\Diamond(B \wedge \Diamond B)$, which is true in w, is a disjunct of some minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}$.

Therefore by conditions ($\star$) and ($\dagger$), and the definition of model M, there is a world $\overline{w} \in W$ such that $\overline{w} \models_{\mathsf{S5}} \Diamond X^{\mathrm{d}}$ and $\overline{w} \not\models_{\mathsf{S5}} \Diamond A^{\mathrm{d}}$. It means that $X \not\models_{\mathsf{AD}_2} A$. $\square$

It seems that for different adaptive logics the following holds: the more mutually consistent consequences in the sense of the lower limit logic of a given adaptive logic, the bigger set of consequences in the sense of the adaptive logic. However, on the basis of Definition 2.4 and Proposition 5 of [19] we have that there are a set X and a formula A, such that $X \nvdash_{\mathsf{ACLuN1}} A$ and $X \nvdash_{\mathsf{ACLuN2}} A$ while $X \vdash_{\mathsf{D}_2} A$, and therefore also $X \vdash_{\mathsf{AD}_2} A$, i.e. treating consequences relations '$\vdash$' as consequences operators 'Cn' we can say that there are X, such that $Cn_{\mathsf{AD}_2}(X) \nsubseteq Cn_{\mathsf{ACLuN1}}(X)$ and $Cn_{\mathsf{AD}_2}(X) \nsubseteq Cn_{\mathsf{ACLuN2}}(X)$. Unfortunately the reverse inclusions — which at first sight seems to be more natural — do not hold. Let's consider the set $X = \{\sim(p \vee q), \sim q \rightarrow p, \sim p, p \vee q\}$. A formula $(p \vee q) \wedge \sim(p \vee q)$ is the minimally consistent CLuN-consequence of the set X, therefore valuations for which the formula $(p \vee q) \wedge \sim(p \vee q)$ is the only inconsistent and valid formula are the only ACLuN1 and ACLuN2-models. So $q \in Cn_{\mathsf{ACLuN1}}(X)$ and $q \in Cn_{\mathsf{ACLuN2}}(X)$. On the other hand $p \wedge \sim p \in Cn_{\mathsf{D}_2}(X)$. A model from Figure 1 is an AD_2-model with respect to the set X, such that $w_1 \models_v \Diamond X^{\mathrm{d}}$. As one can see that $\Diamond q$ is false in the world w_1 of the considered model. Therefore $q \notin Cn_{\mathsf{AD}_2}(X)$. I.e. $Cn_{\mathsf{ACLuN1}}, Cn_{\mathsf{ACLuN2}} \nsubseteq Cn_{\mathsf{AD}_2}$.

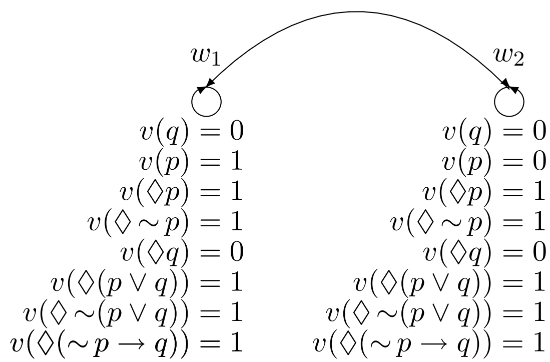

Figure 1. An AD_2-model with respect to the set X. For any other propositional variable B: $v(B) = \varnothing$.

2.1. *Examples of proofs over* AD_2

Example 2.9: *(1)* $\{\sim(p \wedge q), p\} \vdash_{\mathsf{AD}_2} \sim q$.

(2) $\{(p \wedge \sim p) \rightarrow (q \wedge \sim q), \sim(p \vee q), p \vee q, p \vee r\} \vdash_{\mathsf{AD}_2} r$.

Below, we use S5-proofs. The reasoning presented here is adequate since if $\{A_1, \ldots, A_n\} \vdash_{S5} B \vee C$ and $\{A_1 \ldots A_n\} \cup \{C\} \vdash_{S5} D$, then $\vdash_{S5} B \vee (A_1 \wedge \cdots \wedge A_n \to D)$.

Ad 1. Let $X = \{\sim(p \wedge q), p\}$.

1. $\Diamond \sim(p \wedge \Diamond q)$ — assumption
2. $\Diamond p$ — assumption
3. $\Diamond(p \to \sim \Diamond q)$ — 1., extensionality, and the negation of '$\to$'
4. $\Box p \to \sim \Diamond q$ — 3. and distributivity of '$\Diamond$' with respect to '$\to$'
5. $(\Diamond p \to \Box p) \to ((\Box p \to \sim \Diamond q) \to (\Diamond p \to \sim \Diamond q))$ — the transitivity of '$\to$'
6. $(\Diamond p \wedge \Diamond \sim p) \vee ((\Box p \to \sim \Diamond q) \to (\Diamond p \to \sim \Diamond q))$
   5., the disjunctive syllogism, the negation of '$\to$', and de Morgan's law
7. $(\Box p \to \sim \Diamond q) \to (\Diamond p \to \sim \Diamond q)$ — 6. and the fact that '$(\Diamond p \wedge \Diamond \sim p)$' is not a disjunct of any minimally contingent S5-consequence of the set $\Diamond X^{\mathrm{d}}$
8. $\Diamond p \to \sim \Diamond q$ — MP, 7., and 4.
9. $\sim \Diamond q$ — MP, 2., and 8.
10. $\Box \sim q$ — 9. and de Morgan's law
11. $\Diamond \sim q$ — MP, 10., and a substitution of (D)

To finish this proof one has to prove the correctness of step 7. To achieve this, it is enough to show that $(\Diamond p \wedge \Diamond \sim p) \vee (\Diamond q \wedge \Diamond \sim q)$ does not follow from the set $\{\Diamond \sim(p \wedge \Diamond q), \Diamond p\}$ over S5.

In particular, this means that also $\Diamond p \wedge \Diamond \sim p$ is not an S5-consequence of the set of premises. Let's consider a model $\langle\{w\}, \{\langle w, w\rangle\}, v\rangle$, where $v(q) = \varnothing$ and for any other propositional variable B: $v(B) = \{w\}$. Of course in the world w the set $\{\Diamond \sim(p \wedge \Diamond q), \Diamond p\}$ is true but neither $\Diamond \sim p$ nor $\Diamond q$ is true in w. Notice that the application of our adaptive logic was essential, since $\{\sim(p \wedge q), p\} \nvdash_{D_2} \sim q$. To prove this observation let us consider a model $\langle\{w_1, w_2\}, \{w_1, w_2\} \times \{w_1, w_2\}, v\rangle$, where $v(p) = \{w_1\}$ and $v(q) = \{w_1, w_2\}$. In both worlds the set $\{\Diamond \sim(p \wedge \Diamond q), \Diamond p\}$ is true, while the formula $\Diamond \sim q$ is not true anywhere.

Ad 2. Let $X = \{(p \wedge \sim p) \to (q \wedge \sim q), \sim(p \vee q), p \vee q, p \vee r\}$.

1. $\Diamond \sim(p \vee q)$ — assumption
2. $\Diamond(\sim p \wedge \sim q)$ — extensionality and classical version of de Morgan's law
3. $\Diamond \sim p \wedge \Diamond \sim q$ — the distributivity of '$\Diamond$' with respect to '$\wedge$'
4. $\Diamond \sim p$ — 3. and the law of absorption
5. $\Diamond p \vee \Diamond r$ — assumption
6. $\Box \sim p \to \Diamond r$ — 5., de Morgan's law, and the disjunctive syllogism
7. $(\Diamond \sim p \to \Box \sim p) \to ((\Box \sim p \to \Diamond r) \to (\Diamond \sim p \to \Diamond r))$
   the transitivity of '$\to$'
8. $(\Diamond p \wedge \Diamond \sim p) \vee ((\Box \sim p \to \Diamond r) \to (\Diamond \sim p \to \Diamond r))$

the disjunctive syllogism, the negation of '$\rightarrow$', and de Morgan's law

9. $(\Box \sim p \rightarrow \Diamond r) \rightarrow (\Diamond \sim p \rightarrow \Diamond r)$ 8. and the fact, that $(\Diamond p \wedge \Diamond \sim p)$ is not a disjunct of any minimally contingent $\mathsf{S5}$-consequence of the set $\Diamond X^{\mathrm{d}}$

10. $\Diamond r$ MP, 6., 4., and 9.

To finalize the proof we have to prove that $(\Diamond p \wedge \Diamond \sim p)$ is not a disjunct of any minimally contingent $\mathsf{S5}$-consequence of the set $\Diamond X^{\mathrm{d}}$. By Lemma 2.2 the only contingent formulas that potentially could be 'dangerous' as disjuncts of some minimally contingent $\mathsf{S5}$-consequence of the set $\Diamond X^{\mathrm{d}}$ are $\Diamond p \wedge \Diamond \sim p$, $\Diamond q \wedge \Diamond \sim q$ and $\Diamond r \wedge \sim r$. By the laws $\Diamond \sim (p \vee q) \rightarrow \Diamond \sim p \wedge \Diamond \sim q$ and $(\Diamond \sim p \wedge \Diamond \sim q \wedge (\Diamond p \vee \Diamond q)) \rightarrow (\Diamond p \wedge \Diamond \sim p) \vee (\Diamond q \wedge \Diamond \sim q)$ valid for all normal logics we have directly that $(\Diamond p \wedge \Diamond \sim p) \vee (\Diamond q \wedge \Diamond \sim q)$ is an $\mathsf{S5}$-consequence of the set $\Diamond X^{\mathrm{d}}$, while by the assumption $(\Diamond p \wedge \Diamond \sim p) \rightarrow (\Diamond q \wedge \Diamond \sim q)$ we have that a formula $\Diamond q \wedge \Diamond \sim q$ is a minimally contingent $\mathsf{S5}$-consequence of the set $\Diamond X^{\mathrm{d}}$. To finish our reasoning we will show that $(\Diamond p \wedge \Diamond \sim p) \vee (\Diamond r \wedge \Diamond \sim r)$ is not a consequence of the set $\Diamond X^{\mathrm{d}}$. Let's consider a model $\langle \{w_1, w_2\}, \{w_1, w_2\} \times \{w_1, w_2\}, v \rangle$, where $v(p) = \varnothing$, $v(q) = \{w_1\}$, and $v(r) = \{w_1, w_2\}$.The formulas p and $\Diamond p$ are true in neither of the worlds. Thus, the formulas $\sim p$ and $\Diamond \sim p$ are true in both worlds. Notice, that in both worlds w_1 and w_2 the formulas $\Diamond q$ and $\Diamond \sim q$ are true. One can easily see that $\{(\Diamond p \wedge \Diamond \sim p) \rightarrow (\Diamond q \wedge \Diamond \sim q), \Diamond \sim (p \vee q), \Diamond p \vee \Diamond q, \Diamond p \vee \Diamond r\} \nvdash_{\mathsf{S5}} (\Diamond p \wedge \Diamond \sim p) \vee (\Diamond r \wedge \Diamond \sim r)$. Let us mention that neither $\{(p \wedge \sim p) \rightarrow (q \wedge \sim q), \sim (p \vee q), p \vee q, p \vee r\} \vdash_{\mathsf{ACLuN1}} r$, nor $\{(p \wedge \sim p) \rightarrow (q \wedge \sim q), \sim (p \vee q), p \vee q, p \vee r\} \vdash_{\mathsf{ACLuN2}} r$. A formula $(p \vee q) \wedge \sim (p \vee q)$ is the only minimally CLuN-consequence of the set X. One can also easily indicate CLuN-models of the set X, in which $(p \vee q) \wedge \sim (p \vee q)$ is the only inconsistent valid formula, where a formula r at the same time is falsified.

2.2. *A comparison with the logic* $\mathsf{D2}^{\mathsf{r}}$

Joke Meheus has written a very interesting paper in this area ([18]). The main difference, besides other minor differences, is that we use a discussive understanding of '$\wedge$', '$\rightarrow$' and '$\leftrightarrow$'.

We will use the logic $\mathsf{D2}^{\mathsf{r}}$ presented in [18]. We have the following:

Observation 2.10: (1) $\vdash_{\mathsf{AD_2}} \not\subseteq \vdash_{\mathsf{D2^r}}$
(2) $\vdash_{\mathsf{D2^r}} \not\subseteq \vdash_{\mathsf{AD_2}}$

Proof. 1. Indeed, for example $\{\Diamond p, \Diamond \sim p, \Diamond q, \Diamond \sim q\} \vdash_{\mathsf{S5}} \Diamond p \wedge \Diamond q$ and so $\{p, \sim p, q, \sim q\} \vdash_{\mathsf{AD_2}} p \wedge q$, while $\{\Diamond p, \Diamond \sim p, \Diamond q, \Diamond \sim q\} \nvdash_{\mathsf{S5}} \Diamond(p \wedge q)$ and one can easily see that $\{p, \sim p, q, \sim q\} \nvdash_{\mathsf{D2^r}} p \wedge q$.

2. $\{\Diamond q, \Diamond \sim q, \Diamond(\sim p \vee \sim q)\} \vdash_{\mathsf{S5}} \Diamond \sim(p \wedge q)$ so $\{q, \sim q, \sim p \vee \sim q\} \vdash_{\mathsf{D2^r}} \sim(p \wedge q)$, while $\{\Diamond q, \Diamond \sim q, \Diamond(\sim p \vee \sim q)\} \nvdash_{\mathsf{S5}} \Diamond \sim p \vee \Box \sim q$ and one can see that $\{q, \sim q, \sim p \vee \sim q\} \nvdash_{\mathsf{AD_2}} \sim(p \wedge q)$. □

Department of Logic and Semiotics
Nicolaus Copernicus University
Toruń, Poland
E-mail: mnasien@uni.torun.pl

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