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NEGATION, DUALITY AND OPACITY

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Daß aber die Zeichen »p« und »∼p« das gleiche sagen *können*, ist wichtig.
Denn es zeigt, daß dem Zeichen »∼« in der Wirklichkeit nichts entspricht.
However, that the signs ‘p’ and ‘∼p’ *can* say the same is important.
For it shows that nothing corresponds in reality to the sign ‘∼’.
Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, 4.0621

toggle *n.*, 2, h. *Computers.* A key or command that is always operated the same way but has the opposite effect on successive occasions.
Oxford English Dictionary, 2nd edition.

Dla Profesora Piotra Geach, z wyrazami uznania.

Abstract
I argue, along lines first explored by Wittgenstein in the *Tractatus* and elaborated by Peter Geach, (1) that the logical notion of negation may be illuminated by exploiting the notion of a toggle; (2) that the toggle nature of negation may best be appreciated by studying the logical notion of duality; (3) that apparent obstacles to the application of duality due to semantic opacity in intentional contexts may be overcome, and (4) that the result illuminates the connection between duality and the semantics for relevant logics.

Toggles

An electrical device such as a light or a radio typically has a single switch by means of which it is turned on and off. If the device is off, and the switch is operated, the device is turned on. If it is on, then operating the switch turns it off. So whether the device is on or off, a single switching operation takes it to the other, opposed state, whereas switching a second time returns the device to the original state. Operating the switch an odd number of times results in a change of state, whereas operating the switch an even number of times returns one finally to the same state as the original.

Ignoring issues of physical realization, the characteristic features of such a set-up are three: (1) that there are two distinct states; (2) that they are mutually opposed; and (3) that there is a single operation which takes us from either state to the other. I shall call the whole system in such a case a *toggle*. In computing, as the motto quotation indicates, a toggle is something that on repeated operation has the opposite effect on successive occasions. The expression derives from the idea of a toggle switch, like an on/off switch, one operated originally by a lever like a toggle, and snapping back and forth.

Each of the three features of a toggle is essential. If there is only one state, or more than two, we do not have a toggle. A competition marksman may fire a gun standing, lying (prone) or kneeling. These are different states but not poles or opposed states. If there are two states, but they are not mutually opposed, we have no opposition. A leaf is green in the Summer and red in the Fall: these are distinct but not opposed. Finally, if there is no operation taking us back and forth between the two states then we do not have a toggle. Being alive and being dead are two opposed states, but though there is a kind of operation taking one from the living state to the dead — it is known as *killing* — there is no way to return to the living state by repeating the same kind of operation.

There are toggles in many walks of life and some have been used to illustrate the logical. In anything spatial, in geometry or topology, or in geography or astronomy, a line or a path may be understood as having a sense or direction, or as being undirected. The line from A to B is the same as the line from B to A in the undirected sense, but they are opposite lines or directions in the directed sense. The toggle operation in such spatial cases is *turning round* or *reversing direction*.

In the arithmetic of positive and negative integers each number except zero comes paired with an opposite number, $+1$ with -1, $+2$ with -2 and so on. Here we have many opposed pairs. The operation of multiplying by -1 is the general toggle which works for each opposed pair. In integer or real arithmetic the *same* operation takes us from *any* number to its opposite number, and so we may call this operation a *general toggle operation*. Not just any bijection of a set counts as a general toggle: any arbitrary bijection has an inverse but for the bijection to constitute a toggle the inverse function must be the same function as the original, so for example the function $n \to n+1$ is not identical with its inverse, which is $n \to n-1$. Moreover, not just any partition of a set into pairs yields an operation that is intuitively a toggle. There has to be a way in which the objects in each pair are uniformly understood as each others' opposites. We can pair numbers in seventeens, for example, so that $34r + k$ is paired with $34r + 33 - k$, for $0 \leq k \leq 16$, and the invertible operation just switches back and forth across the pairs, but there is no natural way to understand the numbers 37 and 64 or the numbers 10 and 23 as each others' opposites. By contrast $+12$ and -12 are each others' opposites in a natural way, a number and its opposite (in this case: its group- or ring-theoretic additive inverse) being such that their sum is zero.[1] In algebra, a bijective function or operation equal to its inverse is called an *involution*. As the examples show, not every involution is a toggle.

Negation as a toggle

In language the most fundamental kinds of expressions are names and sentences. Intuitively, these differ in their typical semantic roles: names stand for objects, whereas sentences serve to declare, ask about or command things that may be true or false. This much is uncontroversial. However, when we examine standard accounts of the semantic roles or names and sentences we find a less clear distinction. In much semantic theory, names and sentences are alike in that both have denotations and in many theories both have connotations. What these are varies from theory to theory. For Frege, a well-functioning name stands for, denotes or signifies an object, whereas a well-functioning sentence stands for, denotes or signifies one of the two truth-values True and False; a name connotes or has a sense, for which Frege has no specific term, a sentence connotes or has as sense an abstract proposition or what Frege calls a thought. Since Frege takes the truth-values to be objects, sentences are for him a special kind of name. Some theories say names do not have connotation or sense, only denotation, or that only some names (descriptive rather than proper names) have a sense or connotation as well as a reference or denotation. Some theories say sentences connote propositions and denote states of affairs, other theories say that sentences denote truth-values but connote functions from possible worlds to truth-values, or connote sets of possible worlds. The point is not what divides such theories but what unites them: they all have a general relation of denoting which obtains between expressions and entities of some suitable kind, such that the difference between names and sentences lies in the type of entity that they denote, or in addition in the types of entity that they connote.

One of the fundamental stances of Wittgenstein in the *Tractatus* was his opposition to any view according to which sentences denote in anything like the way names do. The original working title for the book was *Der Satz*.[2] In his first extant writing, the *Notes on Logic*, he says flatly, "Propositions

[1] In multiplication among rationals or reals the opposites would be x and $1/x$ where $x \neq 0$ — in this case the opposite is the group- or ring-theoretic multiplicative inverse.

[2] This is not unequivocally translatable into English because *Satz* can be rendered either as 'sentence' or as 'proposition'. Wittgenstein favored the latter, and for the moment we shall follow him.

are not names."[3] Although Wittgenstein struggled with different ways of expressing the fundamental opposition between names and propositions, in all of them he is adamant that propositions do not denote anything — whether truth-values, complexes, states of affairs or facts — as names denote objects. The principal reason for this is that propositions relate to reality not in one way, the denoting way, but in one of two ways, the true way and the false way, and that these ways are opposed to one another. Wittgenstein called this *bipolarity*. "Names are like points", Wittgenstein writes, "propositions like arrows; they have sense (*Sinn*)."[4] The force of the comparison depends on a pun: the German word *Sinn* can mean 'direction'. As we saw above, directions in space or rotations come in opposed pairs forming families of toggle systems, with reversal of direction or sense as the toggle operation. Wittgenstein relies on the familiarity of the idea of direction and reversal of direction to provide a metaphor for the sense of a proposition and negation respectively. But if I am right, the metaphor is fully appropriate in that it reveals not merely a vague similarity but full formal identity of structure: negation is *the* fundamental general toggle operation in language.

Propositions come in opposed pairs: *John loves Mary* is opposed to *John does not love Mary*, *All cats despise dogs* is opposed to *Not all cats despise dogs*, *Leeds United sometimes beat Manchester United* is opposed to *Leeds United never beat Manchester United*. This opposition was one of the first subjects Aristotle examined in his theory of sentences and logic, in *De interpretatione*. The operation which takes us from any proposition to its contradictory opposite may be called 'negation'. Negation is a toggle, it takes us back and forth: when we negate twice, we arrive back where we started: the negation of the negation of a proposition is not a third proposition but the original one again. Names on the other hand have no opposites: they simply name whatever object or objects they denote, and this naming or denoting has no bipolarity about it. Because propositions always come in opposed pairs with negation as the toggle, whereas for names there exists nothing of the sort, propositions (sentences) are not names.

Problems and complications: the variety of truth-bearers

The view just propounded faces a number of difficulties which must be clarified before it can be accepted. The first concerns sentences involving expressions for negation. If the negation of *John loves Mary* is *John does not*

[3] Wittgenstein 1979, 98.

[4] *Tractatus* 3.144.

love Mary, why is the negation of the latter the former rather than the doubly negated sentence *John does not not love Mary*? And why does this not have as its negation the triply negated sentence *John does not not not love Mary*? In general, if '$\sim$' expresses negation, and p is any sentence or proposition, is $\sim\sim p$ the same proposition as p or is it a different one? If they are different, how can this be reconciled with the idea of negation as a toggle? Surely '$\sim$' can be iterated any finite number of times and generate each time a new sentence. If these sentences express ever new propositions, so $\sim\sim p$ is distinct from p, then negation is after all not a toggle, but more like the arithmetic operation of adding one, since each new application of negation takes us to a new proposition in the series.

A quick fix to this problem goes as follows. We should be more careful about distinguishing sentences from propositions. Certainly the *sentence* 'John does not not love Mary' is different from the *sentence* 'John loves Mary', for they consist of partly different words, but they both express one and the same *proposition*. The difficulty is resolved. Negation is a toggle for propositions, since logically equivalent propositions are identical, but not for sentences, which are individuated not by their logical characteristics but by their lexemic constituents and syntactic structure. We should further distinguish negation as an operation on *sentences*, implemented by inserting a negative word like 'not' or some other equivalent modification, from negation as an operation on propositions, which toggles us between a proposition and its unique negation or opposite.

While there is some wisdom in this argument, it is still problematic. If logically equivalent propositions are identical, then there is only one logically true proposition and one logically false proposition. Logically equivalent propositions of mathematics such as Zorn's Lemma, the Axiom of Choice, and the Well-Ordering Principle, to name but three of the many equivalents of the Axiom of Choice, in fact express the same proposition, all appearances to the contrary. One attractive conception of a proposition *qua* sense of a sentence is that propositions are identical if and only if they offer the same computational route to the sentence's truth-value. By this light, $\sim\sim p$ could not be the same proposition as p because there are two additional computational steps in the former case, negating twice, even though the extra steps leave truth-value unchanged. And finally if, as Frege thought, propositional attitude contexts of indirect quotation, such as *John believes that* —, preserve their truth-value when clauses are inserted in the blanks which express the same proposition, then anyone who believes Zorn's Lemma thereby believes the Axiom of Choice etc., again against appearances, and it would appear that there can be no genuine surprises in logic and mathematics. While *John loves Mary* and *John does not not love Mary* are much more obviously equivalent than most pairs of logically equivalent sentences, it is still not ruled out that a person could believe the one and not the other, perhaps through confusion or stubbornness or not having put the two thoughts together. Intuitionists even hold that it is not inconsistent to both believe or assert a double negation and not believe or assert the same doubly unnegated, for example they think all sentences of the form $\neg\neg(p \vee \neg p)$ may be asserted but not all sentences of the form $(p \vee \neg p)$.

As we have seen, when we come to think about what we might mean by 'proposition' there are different tendencies. One tendency is to unify or reduce the numbers of propositions because of their logical equivalence. If we think of a propositions as identical when their truth-conditions coincide, or when they are true in the same possible worlds, then we are led to identify logically equivalent or *cointensional* items. In relevance logic however, propositions may be classically logically equivalent but not identical, for example $(p \mathbin{\&} {\sim}p)$ and $(q \mathbin{\&} {\sim}q)$ for distinct variables, or more prosaically 'It is raining or it is not raining' versus 'It is cold or it is not cold', which intuitively have nothing to do with one another despite both being unconditionally true. Relevantly non-equivalent propositions may both be necessarily true and yet distinct. If we think of propositions as more finely individuated still, so that we may distinguish different logically equivalent or even coentailing propositions because they behave differently in intentional contexts such as *John believes that* —, then we must be far more demanding about the conditions under which two sentences express the same proposition: they must be not only cointensional and coentailing but also synonymous or having the same sense. There is no fixed terminology for the items so individuated, so for present purposes I shall purloin the words 'protension' for propositions as intensions, 'protasis' for propositions as defined under coentailment, and use Frege's word 'thought' for propositions as senses.

All four operands of negation, namely truth-values, protensions, protases and thoughts, are abstract entities defined over a domain of (relative) concreta by equivalence relations of varying strength. The range of concreta from which they derive their being is a varied one covering certain kinds of linguistic token, spoken, written, displayed on screen and otherwise enunciated or uttered, also mental acts and dispositions such as judgements, assumptions and beliefs, and various more devious encodings such as those involving magnetic tapes or sequences of bytes stored temporarily in a chip. In default of a clear delimitation of those entities which can play this role, let us simply give them the functional name of *primary truth-bearers*. The equivalence relations are however much easier to identify: for truth-values it is material equivalence, for protensions it is strict equivalence, for protases it is co-entailment, and for thoughts it is synonymy.

Returning to the idea of negation as a toggle, it can be seen that it is unproblematically a toggle on truth-values, protensions, and protases, and since whatever is coentailing is classically logically equivalent and whatever is classically logically equivalent is also materially equivalent, we do not need three things called 'negation', one for protases, a second for protensions and a third for truth-values, but that one operation will do all three. But the toggle effect of negation does not extend to thoughts. Consider, for any thought p its negation $\sim p$ and its double negation $\sim\sim p$. The double negation of p is materially, classically and relevantly equivalent to p but they are not synonymous, since the former contains two additional occurrences of the negation operator. A thought cannot be a logical complication of itself. Negation on thoughts leads from p to $\sim p$, but we have a choice when looking for the opposite of $\sim p$ where to go next, either back to p or on to $\sim\sim p$. They cannot both be *the* opposite of $\sim p$ because if negation is a toggle it should have only one opposite, not two, yet there seems no reason to prefer p as the opposite to $\sim p$ over $\sim\sim p$, they both have equally good claims, and they express distinct thoughts despite expressing the same protasis and the same protension. Repetition of negation ought to take us back to p if negation were a toggle on thoughts, but it does not.

What we are looking for is a strength of equivalence between primary truth-bearers which is intermediate between coentailment and synonymy, and under which negation is a toggle. The propositional formulas p and $p \,\&\, (p \vee q)$ coentail one another, and equally clearly do not "say the same thing".[5] Yet there is a sense in which $\sim\sim p$ does "say the same thing" as p despite their non-synonymy, as one may similarly think that the members of the pairs $\sim(p \,\&\, q)$ and $\sim p \vee \sim q$, $\sim\forall x.Fx$ and $\exists x.\sim Fx$, $\sim\Box p$ and $\Diamond\sim p$ pairwise "say the same thing", despite containing different symbols. It is perhaps not irrelevant that each symbol of each pair (other than negation) may be used to define the other in a logical system, and our willingness to do this indicates our acceptance that the two sides in such a case *do* "say the same thing".

We may then consider what is the strongest equivalence under which negation remains a toggle. It is not obvious what principles govern such an equivalence. One suggestion, based on work of Richard Angell,[6] is that we use a relation of *analytical containment* to define *analytic equivalence*.[7] Call analytic equivalence (of any expressions, not just sentences) *coindication*. This slightly outstrips the suggestion just made because it makes p coindicative with $p \,\&\, p$ as well as with $p \vee p$, but accepting these as pairwise "saying the same thing" seems to be neither more nor less stringent than accepting $\sim\sim p$ and p as doing so. The exact details may be left aside: the suggestion is that

[5] See Anderson, Belnap and Dunn 1992, 208.

[6] Angell 1989.

[7] See Anderson, Belnap and Dunn 1992, 549. They actually suggest a still slightly stronger equivalence of conjunctive containment as defining "saying the same thing".

we can home in on an equivalence we can agree to be both strong and yet short of the most stringent synonymy. Let us call the abstracta under such an equivalence *propositions* in the case of coindicative sentences and *concepts* elsewhere. By construction, on propositions so understood, negation is a toggle, and each proposition has exactly one negation, even if the sentences expressing it may vary in appearance and constituents.

Duality and dual languages

There is a way to bring out very clearly the distinctness of sentences from names, which is due to Peter Geach, but has its origins in the ideas of Wittgenstein mentioned above. It uses the concept of *duality*. Duality is a concept used widely in algebra, but I am interested in its use in logic. Though first exploited in logic by Ernst Schröder, it has its origin in the De Morgan laws, which in fact go back to Ockham. In logic, duality arises when the concepts of truth and falsity are exchanged in a thoroughgoing manner.[8] The archetype of a pair of dual concepts in logic is clearly conjunction and disjunction. Just as the conjunction of two sentences is *true* if they are both *true*, and *false* otherwise, so the dual of the conjunction of two sentences is *false* if they are both *false*, and *true* otherwise. Disjunction fits this role. Clearly the dual of the dual of any notion is the original notion, since by interchanging truth and falsity twice we get back to the original. However since disjunction is not regarded as the opposite of conjunction, duality, though of period two, is not a toggle. Duality is obviously closely connected with negation, which in classical logic simply switches the truth-values, so it is no surprise that duality and negation interact closely, and the the prime example of this are indeed the De Morgan laws such as $\sim(p \vee q) \dashv\vdash (\sim p \;\&\; \sim q)$. Many familiar logical concepts come in dual pairs: we have seen conjunction and disjunction, in quantification theory we have $\forall$ and $\exists$; the modal operators *necessarily* and *possibly* are duals, as are the deontic operators *obligatory* and *permitted*.

Duality is attached to sentences and expressions which operate directly or indirectly on sentences, but it "bounces off" names. To illustrate we indulge a thought-experiment due to Geach.[9] Imagine a language which is exactly like English in its morphology and syntax but where every sentence has the opposite truth-value of the equiform English sentence. Call this language *Unglish*. Such a language is, appearances perhaps to the contrary,

[8] See Quine 1982 Ch. 12, 79–84.

[9] Geach 1982. The idea is first broached in print not for (or in) English but for (and in) Polish: Geach 1972.

quite thinkable, and it is the dual language to English. What logicians call the dual of a concept is the Unglish translation of an English expression. So if in English we have a conjunction A *and* B, then since the English sentences A and B mean in Unglish what we in English would express by *not* A and *not* B, then since the whole sentence A *and* B means in Unglish what in English we would express as *not* $(A$ *and* $B)$, the way we must translate *and* into Unglish is as *or*, to get the result we need. Conversely, we translate the English word *or* (used in the sense of inclusive disjunction) as *and* in Unglish. Unlike normal bilingual dictionaries, the English-Unglish dictionary goes the same in both directions, since English is also the dual of Unglish.

Proper names in English translate into themselves, because if we wish to deny in one language what we affirm with the same words in the other, we must be talking about the same things. The Unglish translation of a simple English predicate is what looks like its negation, and in general the Unglish translation of any English expression is what looks like its English dual.

Consider a language L and let E be any expression of L. Denote by $*E$ that expression of L whose meaning (in the sense defined by coindication) is that which E receives in its dual language $*\mathsf{L}$. $*E$ is the dual of E in L itself. We write '$=$' between coindicative expressions. For any expression E whatever, $E = **E$. For any sentence S, $*S = {\sim}S$. For any (singular) name N, $*N = N$. Assuming the basic categories of the language are SENTENCE and NAME, the duals of expressions from functor categories may be calculated using the following formula:

$$*(f)(a \ldots z) = *(f(*a \ldots * z)).$$

The dual of variable-binding operators is given by

$$*O\nu_1 \ldots \nu_n(A) = *(O\nu_1 \ldots \nu_n(*A))$$

where $\nu_1 \ldots \nu_n$ are the variables (of whatever category or categories) bound and A is the operator matrix. Note that we take duality not to affect variables, since they can take all values of their category.

Using this schema we can calculate the duals of expressions from standard logic.

Negation: $*({\sim})(S) = *({\sim}(*S)) = {\sim}{\sim}{\sim}S = {\sim}S$, so $*({\sim}) = {\sim}$.
Negation is self-dual.

Conjunction: $(S \ *(\&)\ T) = *(*S \ \& \ *T) = {\sim}({\sim}S \ \& \ {\sim}T) = S \vee T$, so $*\& = \vee$. This is just a more formal way of pointing out what we said in prose before.

Take a simple predication $F(a \ldots z)$:
${\sim}F(a \ldots z) = *(F(a \ldots z)) = *(F)(*a \ldots * z) = *(F)(a \ldots z)$,

so the dual of a predicate is its (predicate) negation. This applies also for logical predicates such as identity and difference, the universal and null predicates.

Consider now propositional quantification:

$$*(\forall)p.p = *(\forall p. *(p)) = \sim\forall p.\sim p = \exists p.p.$$

Hence the dual of the propositional quantifier $\forall$ is the quantifier $\exists$ and vice versa.

Because quantifiers binding any category of variables operate on a sentential matrix (open sentence), an analogous duality will hold for all the universal and particular quantifier pairs of whatever semantic category. A similar analogy holds with conjunction and disjunction for different categories.

In a natural language such as English which, in addition to the basic categories of name and sentence, arguably has a third basic category of common noun, there appears to be a choice as to whether expressions of this category behave under duality like proper names, and are invariant, or like predicates, and so have a dual which is their Boolean negation.[10] Geach advocates the former,[11] so the Geach-dual of the common noun 'cat' would be just 'cat' again, whereas the Boolean dual of 'cat' would be 'non-cat'. Geach's idea goes better with the idea that a common noun denotes severally its associated objects, whereas the Boolean view goes better with the idea of a common noun's being true of its associated objects. Whichever is adopted, duality can be made to work, but the duals of expressions of derived categories are more complex for Boolean duality: for example the Geach-dual of the English quantifier-word *all* is *some* whereas its Boolean dual is *some non-*.

Geach's position on the invariance of common names under duality is coupled with a Wittgensteinian denial that there are genuine complex singular terms.[12] In the light of the ubiquity of such terms as functional expressions in mathematics for example, this is a dubious stance. Fortunately, provided we allow definite description operators to be plural as well as singular, duality for descriptions can be made to work. A description as a term-forming operator is different from a description as a uniqueness quantifier as in Russell: in a sentence of the form *the* F Gs a term-forming operator gives the description narrow scope, governed by the predicate G, and having the form $G(\mathsf{the}\ x[Fx])$, while the Russellian analysis gives the description wide

[10] See Simons 1994.

[11] Geach 1982, 90.

[12] See Geach 1980, 149.

scope and may best be seen as a binary quantifier *THE* $x[Fx; Gx]$. The latter description has the dual $\sim$(*THE* $x[\sim(Fx); \sim(Gx)]$) while the former has the dual the $x[\sim(Fx)]$. The latter has to be meaningful even if its matrix is satisfied non-uniquely, and corresponds roughly to the English *the F or Fs*. Double negations bring us back to the correct result, so that the duals of the whole sentences are the negations of the originals in each case. Following the idea of Whitehead and Russell that functional terms are to be analysed via descriptions, but following not their analysis of descriptions but that of free logic, which allows descriptions as genuine terms, we are able to apply duality to all functional expressions, even when the functions in question are partial. So we can dualize mathematics without extensive rewriting, endorsing the view that there can be a whole dual language.

Not all complex terms are self-dual however. Whether they are depends on their arguments. Characteristic functions associated with predicates are not. As a generalization of these, consider the type of functor of two nominal arguments and one sentential argument char$(a, b)(p)$ which is defined as

$$\text{char}(a, b)(p) = a \text{ if } p \text{ and } \text{char}(a, b)(p) = b \text{ if } \sim p.$$

For obvious reasons this cannot be self-dual if $a \neq b$; rather the dual of char(a, b) is char(b, a), which is just as we'd expect, since char(a, b) as it were arbitrarily assigns two objects to do the work of Frege's two truth-values, and duality consists precisely in swapping the roles of truth and falsity so we'd naturally expect a and b to swap their locally assigned roles under duality.

A logical language does not need to be bivalent (have just the two values *true* and *false*) for the notion of duality still to make sense. A three-valued logic with a neutral value still allows switching of the non-neutral truth-values, leaving the neutral one alone. For a many-valued logic with either finitely or infinitely many values in the interval $[0, 1]$, ordered by the natural arithmetic ordering, duality arises through the interchange of values v and $1 - v$, for any $v \in [0, 1]$. We shall look at an interesting four-valued case in the last section.

Having seen just how differently names and sentences behave under duality, the temptation to think that sentences are a funny kind of name, or that they stand for a funny kind of thing (truth-value, state of affairs), is reduced almost to nothing. Negation does not correspond to anything in the world, because the negation sign could not simply vanish under double negation, nor could a sentence with a negation sign in it mean opposite things in dual languages. That does not mean (*pace* Wittgenstein) that we cannot nominalize sentences to get *that*-clauses, but it does mean that we cannot take seriously the idea that such clauses should stand for anything. The grass-roots use of the sentential is proposition-expressing, and nominalization does not alter that. Nominalization is what it seems: a syntactic adapter, a grammatical device bred of the syntactic poverty of natural languages in cultivating higher-order functors, and allowing us to say things that otherwise would not easily pass our lips.

Duality and opacity

Semantically opaque contexts are those induced by operators such as modal operators and verbs of propositional attitude where some usual case of substitution of expressions *salva veritate* breaks down. Thus modal contexts are opaque to expressions which are merely coextensional, while intentional contexts such as *John believes that* — are opaque to cointentional and coentailing expressions. This creates a problem for duality as follows. Represent a sentence like 'a believes that p' as '$\mathrm{B}ap$'. Then by principles of duality

$$\sim\mathrm{B}ap = *(\mathrm{B}ap) = *\mathrm{B} * a * p = *\mathrm{B}a\sim p$$

In line with our treatment of the duals of predicates as their negates it would seem that we might treat the dual of 'believes' as 'does not believe'. But this gives us the wrong result, for not believing that p is quite different from not believing that $\sim p$. Alternatively, if we treat 'believes' as self-dual, we get that believing that $\sim p$ and not believing that p are the same thing, but they are not: an agnostic who does not believe there is a God is different from an atheist who believes there is not a God.

It may be suggested, following Frege, that we need to treat the clause p governed by a verb of propositional attitude as having an oblique meaning, different from its meaning in standard contexts, whether extensional or intensional. It is by no means clear how to do this, and in any case the idea that a sentence has different meanings according to context is one which we should avoid if possible or at least minimize its application wherever we can. In this case we can avoid treating the embedded clause p differently provided we treat the dual of *believes* as *not believes not*, that is[13]

$$*\mathrm{B}ap = \sim\mathrm{B}a\sim p, \text{ so that } *(\mathrm{B}ap) = \sim\mathrm{B}a\sim\sim p$$

In view of the manifest ability of human beings sincerely to believe contradictions, it may seem that this is still insufficient, because someone who does not observe the equivalence of p and $\sim\sim p$, or who like an intuitionist does

[13] Cf. Geach 1982, 92–3, where Geach is endorsing the view of Prior 1971, 16 ff., that a sentence in a belief context is still functioning as a sentence.

not believe in the equivalence, may believe that p and yet not believe that $\sim\sim p$, or vice versa. So is duality after all inapplicable to intentional contexts? There are two ways to avoid this conclusion. One is to consider a new verb of propositional attitude, *contrabelieving*, which applies only to negative propositional arguments, so that to contrabelieve that $\sim p$ is precisely to believe that p, and then to determine that the dual of a *believes that* p is a *does not contrabelieve that* $\sim p$, which gives the right result. However, this leaves contrabelieving a positive proposition as something meaningless, and seems suspiciously *ad hoc*.

The other and preferable solution is to recall the finely balanced notion of *proposition* mooted at the end of section 3. According to this, p and $\sim\sim p$ are one and the same proposition, so that believing p and believing $\sim\sim p$ are the same. This can be reconciled with someone's openly adopting and evincing different attitudes in respect of the two sentences: because p and $\sim\sim p$ are not *synonymous*, the two sentences express different thoughts, and it can fail to be patent to the person that they are taking up contradictory attitudes to one and the same *proposition*. Duality, as we saw, does not apply to thoughts any more than it does to sentences. An attitude which, unlike belief with respect to propositions, fails to sustain duality, must then contain some element of quotation, or be sensitive to replacement of non-synonymous expressions for concepts.

Since the expression *the negation of* seems to indicate the presence of a negation sign, and give rise to precisely the difficulties with which we have been struggling, it is often better to refer to the dual of a proposition not as its negation but as its *contradictory*. We may then characterize the dual of believing a proposition as not believing its contradictory.

Two by two equals four: duality, De Morgan and relevance

All humans are fallible and their beliefs are therefore typically both incomplete and inconsistent. With respect to an arbitrary proposition p a person a might believe it or disbelieve it, that is, believe its contradictory, but because of incompleteness she might fail to believe either it or its contradictory, and because of inconsistency she might both believe and disbelieve it. So the following four combinations are possible:

[1] a believes that p and does not disbelieve that p
[2] a disbelieves that p and does not believe that p
[3] a neither believes nor disbelieves that p
[4] a both believes and disbelieves that p

Consider a given person a at a certain time and the propositions she believes.

Let us not idealize to the extent of supposing she is consistent or complete, nor that she believes all the consequences of any proposition she does believe: real people are not like that. The collection of all propositions believed by this person is however, let us idealistically suppose, a determinate set: a proposition is either in it or not (i.e. we are ignoring degrees of belief). Call this set a's *belief state*. Of course the belief state will be typically unlike a genuine world-describing set of propositions, which has to be consistent and complete. Of any given proposition, its status with respect to a's belief state must logically be exactly one of [1]–[4].

Consider now how a's belief state behaves under duality. Since the dual of believing a proposition is not believing its contradictory, the dual state of a's beliefs contains as positive beliefs all the propositions that a does not disbelieve and as disbeliefs all the propositions that a does not positively believe. The net result is to leave propositions having status [1] and [2] unchanged in the dual state, but to exchange statuses [3] and [4]. A proposition that a neither believes nor disbelieves is one she dually both believes and disbelieves, and vice versa. Of course dual belief is not a status that will typically make a difference to a's inner life, because most propositions are neither believed nor disbelieved as a result of simply not having any idea at all about their subject matter.

a's belief state $\mathrm{BS}(a)$ and dual belief state $*\mathrm{BS}(a)$ sustain the following generalization about a proposition and its dual, contradictory, or negation:

[$*$] for any p, $\sim p \in \mathrm{BS}(a)$ if and only if $p \notin *\mathrm{BS}(a)$.

There are two salient features of this fact. The first is that if a person's belief state is "angelic", that is, complete and consistent, then $*\mathrm{BS}(a) = \mathrm{BS}(a)$ and the fact [$*$] reduces to the truth-clause for negation in ordinary bivalent semantics. The second is that the fact [$*$] characterizes a notorious construction in the semantics for relevant logics, involving an operator known as the Routley Star.[14] In this semantics, the negation of a proposition p is true at a world w if and only if its p is false not at that world but at a related world $*w$ called the starred world of w. The star operator, although it works perfectly to give the right results in the semantics for relevant logic, is often accused of lacking motivation. However what the Routleys call *set-ups*, the inconsistent and incomplete worlds needed for that semantics, are precisely like belief-states as we have characterized them, so that the star operator may be construed as duality applied to beliefs and belief states, and in this regard may be considered wholly natural. Further the four statuses [1]–[4] may be interpreted in terms of a non-standard account of truth and falsity. We call

[14] After Routley and Routley 1972.

a proposition p true in $\mathrm{BS}(a)$ iff $p \in \mathrm{BS}(a)$, and relabel the four possible truth-statuses of p in $\mathrm{BS}(a)$ as

T	true and not false	(as [1])
F	false and not true	(as [2])
N	neither true nor false	(as [3])
B	both true and false	(as [4]).

This four-valued semantics is characteristic for relevant implication between truth-functions in the same way that two-valued semantics using the first two classical values is characteristic for classical propositional logic. The four values can provide an alternative semantics for relevant logic in general.[15]

What this unexpected link-up shows is that duality is a concept not only applicable but definitely at home outside the narrow confines of bivalent and classical logic, and that it can be used to shed light in such areas. The concept of negation supported by the clause [$*$] in either a two-valued or a four-valued semantics for relevant logic is what is known as *De Morgan negation*, and there are many reasons for regarding this as the proper way to understand negation when we leave the logical safe haven of truth-functions.[16] This brings us full circle, since duality is the concept which arises in considering the De Morgan laws for truth-functions. Indeed one of the original semantic approaches to De Morgan negation in relevant logic by Michael Dunn[17] represented propositions as pairs (A^+, A^-) where A^+ represents the positive information a proposition gives and A^- represents its negative information. Unsurprisingly, the negation of a proposition is represented simply as the

[15] The four values are mooted as suitable for a relevant logic for databases in Belnap 1972: the work is incorporated (as §81) in the definitive Anderson, Belnap and Dunn 1992, as is much of the work of Routley, Meyer and Dunn on the semantics of relevant logics. Dunn 1999 provides a useful broad survey of prior work in the semantics of negation, where we learn (34) that the idea of the four values goes back to Sanjay, an Indian logician working before the 6th C BCE. In Simons 1989 §5 I show that similar ideas were evolved independently by Meinong, who however stops short at three values, though he should have seen the fourth.

[16] De Morgan negation takes T to F and vice versa, just like classical negation, but it leaves B and N propositions as they are in status. The rival Boolean negation switches the statuses B and N. Boolean negation validates the "irrelevant" inference $(p \,\&\, \textit{not}\, p) \vdash q$, which De Morgan negation does not. Restall 1999 rehearses reasons for thinking that Boolean negation is not even coherent as a concept of negation.

[17] Dunn 1966.

pair (A^{-}, A^{+}), so negation for Dunn's proposition-surrogates is a polarity-reversing toggle, just as Wittgenstein said it was.

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