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A LOGICAL APPROACH TO QUALITATIVE REASONING WITH 'SEVERAL'

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Abstract
We examine logical systems, with generalized quantifiers, for expressing and reasoning about (some versions of) 'several'. The primary motivation is a qualitative approach to assertions and arguments involving vague notions, found in ordinary language and in some branches of science. Intuitions underlying (versions of) 'several' are made precise by means of families of subsets. This gives a conservative extension of classical first-order logic, with which it shares various properties. Its sorted version captures relative notions with appropriate behavior.

1. *Introduction*

In this paper we examine logical systems,with generalized quantifiers, for the precise treatment of assertions involving some versions of 'several' (or 'many'). We wish to express such assertions and reason about them in a precise manner. Here, we will concentrate on some specific reasonable versions of 'several'.

Assertions and arguments involving vague qualitative notions, such as 'generally', 'several', 'many', etc., occur often, not only in ordinary language, but also in some branches of science. For instance, one often encounters assertions such as "Bodies 'generally' expand when heated" and "Birds 'generally' fly". Somewhat vague terms, such as 'likely', 'prone', etc., are frequently used in everyday language. More elaborate expressions involving 'propensity' are often used as well. For instance, a physician may say that a patient's genetic background indicates a certain 'propensity', which makes him or her 'prone' to some ailments.

We wish to express assertions, such as "Several people love music", and reason about them in a formal manner. To *express* such 'several' assertions formally, we introduce the new operator ∇, and express "Several people are music lovers" by $\nabla v M(v)$. To give precise *meaning* to such assertions we extend the usual notions, by providing a family $\mathcal{C}$ of 'sizable' sets, and stipulate that $\nabla v M(v)$ means that the set $\{p \in P : M(p)\}$, where P represents a set of people, is in the family $\mathcal{C}$ as a rigorous counterpart for "the set of people that are music lovers is a 'sizable' set". To *reason* about such 'several' assertions in a formal manner, we will set up a deductive system by extending the classical first-order predicate calculus.

Our logic belongs to a family of closely connected systems with generalized quantifiers [16, 3] for qualitative reasoning about some vague notions. A logic with various generalized quantifiers, for notions such as 'many', 'few', etc., has been suggested as appropriate to treat quantified sentences in natural language [2]. Also, some traditional square-of-opposition relations among 'few', 'many', and 'most' have been analyzed [17]. Here, we concentrate on logical systems with generalized quantifiers for capturing some specific reasonable meanings of 'several' (or 'many')[1]. We aim at extending a treatment suggested in [9, 4] along the lines pursued in ultrafilter logic [5, 22, 23, 24].

The structure of this paper is as follows. We begin by motivating the usage of families of sets for capturing some intuitive ideas of 'several'. Next, we consider a basic system for reasoning about these versions of 'several': we introduce our (unsorted) logic for 'several', in section 3, and examine some properties, such as completeness, in section 4. More interesting situations, however, require assertions relative to several universes, involving "several birds" and "several penguins", for instance, which we take up in section 5, where we motivate ideas concerning 'relative several' and introduce our sorted framework for reasoning about versions of 'several' relative to various universes. Finally, section 6 presents some concluding remarks.

2. *Meanings of 'Several'*

In this section we discuss, trying to explain and justify, some issues in the precise treatment of assertions involving 'several'. We shall focus mainly on the usage of families of sets for capturing the intended meaning of such assertions by analyzing some basic intuitions and their underlying presuppositions.

[1] Other meanings as well as notions with similar meanings, like 'most', may be expected to be captured by related logics, such as filter logic [4, 9, 25]. We will comment some more on this point in the concluding section.

2.1. *Some accounts for 'several'*

Various possible interpretations seem to be associated with somewhat vague notions of 'several'. We shall now examine some reasonable ones.

Consider an assertion of the form "Several objects have a given property". How is one to understand this assertion? What would be the possible grounds for accepting it? We shall now examine some answers to these questions stemming from possible accounts for 'several'.

Some accounts for 'several' try to explain it in terms of size.

For instance, consider the assertion "Several Viennese like waltz". A size-based account for it might be "The Viennese that like waltz form a 'large' set", in the sense that their number is above, say, 1 million[2].

Such accounts of 'several' try to explicate "several objects have a given property" as "the objects having this property form a 'large' set", i.e., a set having 'high' cardinality, where 'high' is understood as above a stipulated threshold. Alternatively, we may say that the set of exceptional objects — those failing to exhibit this property — is 'small', with similar sense.

For instance, consider the assertion "Several natural numbers fail to divide sixty". One may interpret, and explain, it by regarding it as asserting that "the divisors of sixty form a 'small' set", where 'small' is understood as finite. Similarly, one would understand the assertion "Several real numbers are irrational" as "the rational reals form a 'small' set", with 'small' now taken as (at most) denumerable.

Other accounts for 'several' may rely on relative frequency or probability.

For instance, consider the assertion "Brazilians generally like soccer". A relative-frequency account for it may be "The Brazilians that like soccer form a 'likely' portion", with more than, say, 77% of the population. A size-based account for it might be "The Brazilians that like soccer form a 'sizable' set", in the sense that their number is above, say, 130 million[3].

The accounts based on size and on relative frequency are similar, in that they try to reduce 'several' to a measurable aspect and a given threshold. They seek to explicate "several objects have property φ" as "the objects having φ form a 'likely' (or 'large') set", i.e., a set having 'high' relative frequency (or cardinality), where 'high' is understood as above the given threshold (capturing some idea of 'high').

[2] Here, we use 'Viennese' as "inhabitant of Vienna". Notice that this threshold may depend on the person. Other persons may be inclined to use other thresholds and accept that a set of Viennese is 'large' when its size exceeds 1.2 million or, say, 0.8 million.

[3] Brazil has about 170 million inhabitants. Here, we use 'Brazilian' as "inhabitant of Brazil".

These two metric accounts, however, differ in one important aspect, namely the behavior of the sets with 'several' elements under permutations of the universe. On the one hand, the 'large' account — cardinality above a given threshold — clearly fails to distinguish sets with the same cardinality: they are all either 'large' or non-'large'. In contrast, sets with the same size may very well have distinct probabilities[4].

Still other accounts for 'several' (with more qualitative character) are also possible[5]. As more neutral names encompassing these, distinct, notions, we will prefer to use 'sizable' in lieu of 'large' or 'likely', and, accordingly 'negligible' for 'unlikely' or 'small'.

The previous terms are somewhat vague, the more so with the new ones. Nevertheless, they present some advantages. On the one hand, they do not resort to a somewhat arbitrary threshold. On the other hand, they have a wider range of applications, stemming from the interpretation of 'sizable' as carrying considerable weight or importance.

As an example of a possible interpretation of 'sizable', imagine that a socialite visiting Hollywood, and eager to attend interesting parties, receives the advice: "Interesting parties are those attended by celebrities". This advisor is understanding as 'sizable' sets of guests those including some celebrities, say Madonna or studio executives. Notice that, as our examples suggest, the notion of 'sizable' is relative to the situation or person[6].

2.2. *'Several' and families of sets*

One usually understands "Several birds fly" as "The flying birds form a 'sizable' set". This indicates that the intended meaning of "Several objects have a given property" can be rendered as "The set of those objects having this property is sizable", in the sense of belonging to a given family of sizable sets.

[4] Indeed, any infinite universe V can be partitioned as the union of two sets X and Y with the same cardinality as V (by resorting to the axiom of choice); so V, X and Y cannot have all the same probability (since Prob(X)+Prob(Y)=Prob(V)=1 we cannot have Prob(X)=Prob(Y)=Prob(V)), even though they have the same size. For a finite universe, it suffices to consider a non-uniform probability distribution.

[5] For instance, accounts based on dense sets in a given topology [4, 9, 25].

[6] As an example illustrating the relative character of 'sizable' as carrying considerable importance, consider two sets: one consisting of thirty birds, and another one consisting of a couple of elephants. The Zoo director is likely to consider them equally important. An ornithologist, however, might rank the former as more important, whereas a truck driver in charge of transporting them would probably give more attention to the latter. So, a smaller set may be more important than a larger set, or just as important.

The relative character of 'sizable' is embodied in the family of sizable sets, which may vary according to the situation. Such families, however, can be expected to share some general properties, if they are to be appropriate for capturing notions of 'sizable', such as 'large' or 'likely'.

Some properties that a family of sizable sets may, or may not, be expected to present are illustrated in the next example.

Example (Brazilians and shaving) Consider the universe of Brazilians, understood as "inhabitants of Brazil", and imagine that one accepts the following two assertions.

1. "Several Brazilians have their beards shaved",
2. "Several Brazilians shave their legs".
   In this case, one would probably accept also the following assertion:
3. "Several Brazilians have their beards shaved *or* sport a moustache".
   This, however, does not seem to be the case with
4. "Several Brazilians have their beards shaved *and* shave their legs".
   An explanation for not accepting assertion 4 is as follows[7]. The "Brazilians that have their beards shaved" are generally males, whereas the "Brazilians that shave their legs" are generally females. So, the "Brazilians that have their beards shaved *and* shave their legs" form a very small fraction of the population. □

The preceding example illustrates the following ideas:

- if H is a subset of D and H has 'several' elements, then D also has 'several' elements;
- even though both H and L have 'several' elements, their intersection $H \cap L$ may fail to have 'several' elements.

So, a family of sizable sets — of those having 'several' elements — is expected to be closed under supersets, but *not* under intersection.

We can now postulate some reasonable properties of a family $\mathcal{C} \subseteq \wp(V)$ of sizable sets (in the sense of having 'several' elements) of a universe V[8].

The first property is, as suggested above, closure under supersets.

($\supseteq$) $Y \in \mathcal{C}$ whenever $Y \supseteq X \in \mathcal{C}$ {"Supersets of sizable sets are sizable"}.

A family $\mathcal{C} \subseteq \wp(V)$ closed under supersets will be called *upward closed*.

[7] The reason for accepting assertion 3 should be clear.

[8] Similar postulates for 'many' are suggested in [2]. For instance, our postulates ($\supseteq$) and ($\emptyset$) below are versions of postulates SP2 and SP6, respectively, in [2 (p. 208, 209)].

The other two properties concern the non-triviality of our notion of 'sizable': the existence of sizable and non-sizable sets[9].

(V) $V \in \mathcal{C}$ {"The universe V is sizable"}.

($\emptyset$) $\emptyset \notin \mathcal{C}$ {"The empty set $\emptyset$ is not sizable"}.

A family $\mathcal{C} \subseteq \wp(V)$ with properties (V) and ($\emptyset$) will be called *proper*.

Thus, we are suggesting that a family of sizable sets — of those having 'several' elements — is a proper upward closed family, but not necessarily a filter[10]. More definitely, each family of sets with 'several' elements is proper and upward closed, and, conversely, each notion of 'several' gives rise to an upward closed family, which will be proper if the notion is non-trivial.

Under the light of these observations, the interpretation of "several objects have property φ" as "the set of objects having property φ is sizable" can be seen to amount to "the set of objects having φ belongs to a given proper upward closed family".

Thus, generalized quantifiers, denoting families of sets [2 (p. 163)], now appear natural to capture our notion of 'several'.

3. *Logic for 'Several'*

Our logic for 'several' extends classical first-order logic by a generalized quantifier ∇, whose behavior will be seen to be intermediate between $\forall$ and $\exists$. We now examine this logic $\mathcal{L}_{\omega\omega}(\rho)^{s}$ — its syntax, semantics and axiomatics — and illustrate some of its features. In examples, we may paraphrase "several objects" by "objects generally".

We use familiar logical concepts and notations, as in [1, 6, 7, 8, 20]. We consider a fixed denumerably infinite set V of for variables. Given a signature (logical type) ρ, with repertoires of symbols (distinct from the variables) for predicates, functions and constants, we let $\mathsf{L}(\rho)$ be the usual first-order language (with equality $\simeq$) of signature ρ, closed under the propositional connectives and the classical quantifiers $\forall$ and $\exists$.

3.1. *Syntax of* ∇

We will use $\mathsf{L}^{\nabla}(\rho)$ for the extension of the usual first-order language $\mathsf{L}(\rho)$ obtained by adding the new operator ∇.

[9] In view of postulate ($\supseteq$), the existence of some sizable set is equivalent to the universe being sizable, and the existence of a non-sizable set is equivalent to the empty set being non-sizable.

[10] Filters may be appropriate for other, stronger, notions [9, 25].

The formulas of $\mathsf{L}^{\nabla}(\rho)$ are built by the usual formation rules and the following new variable-binding *formation rule* for generalized formulas:

(∇) for each variable $\mathrm{v} \in \mathrm{V}$, if φ is a formula in $\mathsf{L}^{\nabla}(\rho)$ then so is $\nabla \mathrm{v}\varphi$.

We shall also employ the notation $\varphi[\mathrm{v} := \mathrm{t}]$ for the result of substituting term t for all the free occurrences of variable v in formula φ, which we may simplify to $\varphi(\mathrm{t})$, when safe. Other syntactic notions, such as sentence, etc., can be appropriately adapted.

The next example illustrates the expressive power of such languages.

Example (Expressive power of ∇) Consider signature λ consisting of the binary predicate L (with $\mathrm{L}(\mathrm{x,y})$ standing for x loves y). We can then express some assertions by means of first-order sentences; for instance, “Everybody loves somebody” by $\forall \mathrm{x} \exists \mathrm{y}\ \mathrm{L}(\mathrm{x,y})$ and “Somebody loves everybody” by $\exists \mathrm{x} \forall \mathrm{y}\ \mathrm{L}(\mathrm{x,y})$.

Some assertions expressed by means of the quantifier ∇ are as follows.

- “Several people love somebody” by $\nabla \mathrm{x} \exists \mathrm{y}\ \mathrm{L}(\mathrm{x,y})$.
- “Somebody loves several people” by $\exists \mathrm{x} \nabla \mathrm{y}\ \mathrm{L}(\mathrm{x,y})$.
- “Everybody loves several people” by $\forall \mathrm{x} \nabla \mathrm{y}\ \mathrm{L}(\mathrm{x,y})$.
- “Several people love everybody” by $\nabla \mathrm{x} \forall \mathrm{y}\ \mathrm{L}(\mathrm{x,y})$.
- “People generally love each other” (in the sense “Several people love several people”) by $\nabla \mathrm{x} \nabla \mathrm{y}\ \mathrm{L}(\mathrm{x,y})$.

These sentences indicate several ways of using the new quantifier ∇. □

3.2. *Upward closed semantics for ‘several’*

The semantic interpretation for our logic of ‘several’ is provided by enriching first-order structures with upward closed families of sets and extending the usual definition of satisfaction to the generalized quantifier ∇.

An *upward closed structure* $\mathfrak{M}^{\mathcal{C}} = (\mathfrak{M}, \mathcal{C})$ for signature ρ consists of a first-order structure $\mathfrak{M}$ for signature ρ together with a proper upward closed family $\mathcal{C}$ over the universe M of $\mathfrak{M}$.

We extend the usual definition of *satisfaction* of a formula φ in a structure under an assignment $\mathfrak{s} : \mathrm{V} \to \mathrm{M}$ to variables as follows

($\vDash^{\nabla}$) for a generalized formula $\nabla \mathrm{v}\varphi$, we define $\mathfrak{M}^{\mathcal{C}} \vDash \nabla \mathrm{v}\varphi[\mathfrak{s}]$ iff the set $\{\mathrm{m} \in \mathrm{M} : \mathfrak{M}^{\mathcal{C}} \vDash \varphi[\mathfrak{s}(\mathrm{v} \mapsto \mathrm{m})]\}$ belongs to the upward closed family $\mathcal{C}$; where, as usual, $\mathfrak{s}(\mathrm{v} \mapsto \mathrm{m})$ is the assignment agreeing with $\mathfrak{s}$ on every variable but v, and $\mathfrak{s}(\mathrm{v} \mapsto \mathrm{m})(\mathrm{v}) = \mathrm{m}$.

As usual, satisfaction of a formula depends only on the realizations assigned to its symbols[11]. Thus, satisfaction for purely first-order formulas

[11] In particular, satisfaction of a formula hinges only on the values assigned to its free variables.

(without ∇) does not depend on the family $\mathcal{C}$, i.e., for a formula φ of $\mathsf{L}(\rho)$, we have $\mathfrak{M}^{\mathcal{C}} \models \varphi[\mathfrak{s}]$ iff $\mathfrak{M} \models \varphi[\mathfrak{s}]$.

Other semantic notions, such as reduct, model ($\mathfrak{M}^{\mathcal{C}} \models \Gamma$), validity, etc., are as usual.

Also, the notion of *upward closed consequence* is as expected: $\Gamma \models^{\mathcal{C}} \tau$ iff $\mathfrak{M}^{\mathcal{C}} \models \tau$ whenever $\mathfrak{M}^{\mathcal{C}} \models \Gamma$.

The behavior of the new generalized quantifier ∇ is easily seen to be intermediate between those of the classical quantifiers. The behavior of iterated ∇'s, however, contrasts sharply with the commutativities of each classical $\forall$ and $\exists$: the formulas $\nabla y \nabla x\, \varphi \to \nabla x \nabla y\, \varphi$ fail to be valid.

3.3. *Axiomatics of upward closed logic*

We will now formulate a deductive system for our logic by adding schemas, coding properties of upward closed families of sets, to a calculus for classical first-order logic.

We set up a deductive system for upward closed logic by taking a sound and complete deductive calculus for classical first-order logic, with Modus Ponens as the sole inference rule (as in [7 (p. 104)])[12], and extending its set $\mathrm{A}(\rho)$ of axiom schemas by adding a set $\mathrm{B^s}(\rho)$ of new axiom schemas (coding properties of upward closed families), to form $\mathrm{A^s}(\rho) = \mathrm{A}(\rho) \cup \mathrm{B^s}(\rho)$.

The set $\mathrm{B^s}(\rho)$ of new axiom schemas consists of all the generalizations of the following four schemas (where φ, ψ and θ are formulas of $\mathsf{L}^{\nabla}(\rho)$):

$[\forall\nabla]$: $\forall v \varphi \to \nabla v \varphi$;
$[\to \nabla]$: $\forall v(\psi \to \theta) \to (\nabla v \psi \to \nabla v \theta)$;
$[\nabla\exists]$: $\nabla v \varphi \to \exists v \varphi$;
$[\nabla\nu]$: $\nabla v \varphi \to \nabla u \varphi\,[v := u]$, for a new variable u[13].

These four axiom schemas express properties of upward closed families by means of ∇, the last one covering a version of alphabetic variants[14].

Thus, upward closed derivability amounts to first-order derivability from the upward closed schemas, more precisely:

[12] Using such a system is convenient, but not strictly necessary. In it, generalization becomes a derived rule [8 (p. 109)].

[13] By a new variable we mean one not occurring in formula φ.

[14] Schemas similar to $[\to \nabla]$ and $[\nabla\nu]$ appear in [12]. Also, one may replace the schemas $[\forall\nabla]$ and $[\nabla\exists]$ by the axioms $\nabla vv \simeq v$ and $\neg\nabla v \neg v \simeq v$, much as in the case of topological logic [19].

$$\Gamma \vdash^{s} \varphi \text{ iff } \Gamma \cup \mathrm{A}^{s}(\rho) \vdash \varphi \qquad (\vdash^{s} \,\&\, \vdash).$$

In particular, we have monotonicity of $\vdash^{s}$: $\Gamma \cup \Delta \vdash^{s} \varphi$, whenever $\Gamma \vdash^{s} \varphi$.
All the formulas $\forall \mathrm{v}(\psi \leftrightarrow \theta) \to (\nabla \mathrm{v}\psi \leftrightarrow \nabla \mathrm{v}\theta)$ are provable. As a result, we have substitutivity of equivalents: if $\Gamma \vdash^{s} \psi \leftrightarrow \theta$ then $\Gamma \vdash^{s} \nabla \mathrm{v}\psi \leftrightarrow \nabla \mathrm{v}\theta$.

Example (People in conflict). Consider the following facts about a universe of people.

1. “People generally oppose those in conflict with any one with whom they sympathize”, in the sense “Several people oppose several people in conflict with any one with whom they sympathize”, expressed by $\nabla \mathrm{x}\forall \mathrm{y}\nabla \mathrm{z}[\mathrm{S}(\mathrm{x},\mathrm{y}) \wedge \mathrm{K}(\mathrm{z},\mathrm{y}) \to \mathrm{O}(\mathrm{x},\mathrm{z})]$.
2. “Several people sympathize with Charles”, expressed by $\nabla \mathrm{y}\mathrm{S}(\mathrm{y},\mathrm{c})$.

Then, one can infer the sentence $\nabla \mathrm{x}\nabla \mathrm{z}[\mathrm{K}(\mathrm{z},\mathrm{c}) \to \mathrm{O}(\mathrm{x},\mathrm{z})]$ {expressing “People generally oppose those in conflict with Charles”}. □

Other usual deductive notions, such as (maximal) consistent sets, witnesses, conservative extension [7, 20], can be appropriately adapted.

4. *Upward Closed Logic*

We shall now establish some properties of our logic for ‘several’, including soundness and completeness of our deductive system with respect to upward closed consequence.

4.1. *Soundness*

We first examine the soundness of our deductive system with respect to consequence. As usual, soundness is easily established.
Indeed, the schemas in $\mathrm{B}^{s}(\rho)$ code properties of upward closed families, so they hold in all upward closed structures[15]. Thus, as Modus Ponens preserves validity, we have soundness of our deductive system with respect to consequence: $\vdash^{s} \subseteq \models^{\mathcal{C}}$.

[15] Clearly, the schemas $[\forall\nabla]$, $[\nabla\exists]$ and $[\to \nabla]$ code properties of upward closed families. As for $[\nabla\nu]$, if variable u does not occur in φ, we have $\mathfrak{M}^{\mathcal{C}} \models \varphi[\mathrm{v} := \mathrm{u}][\mathfrak{s}]$ iff $\mathfrak{M}^{\mathcal{C}} \models \varphi[\mathfrak{s}(\mathrm{v} \mapsto \mathfrak{s}(u))]$, so $\mathfrak{M}^{\mathcal{C}} \models \nabla \mathrm{v}\varphi[\mathfrak{s}]$ iff $\mathfrak{M}^{\mathcal{C}} \models \nabla \mathrm{u}\varphi[\mathrm{v} := \mathrm{u}][\mathfrak{s}]$.

4.2. *Completeness*

To show our deductive system complete with respect to consequence, we can adapt Henkin's well-known proof for classical first-order logic [11, 7, 20]. The main point is providing an upward closed family, which we can do by means of witnesses. We proceed to outline how this can be done.

Given a consistent set Γ in $\mathsf{L}^{\nabla}(\rho)$, we can extend it to a maximal consistent set Σ in language $\mathsf{L}^{\nabla}(\rho \cup \mathrm{C})$, with witnesses for the existential sentences of $\mathsf{L}^{\nabla}(\rho \cup \mathrm{C})$ in set C of new constants with $\mid \mathrm{C} \mid \leq \mid \mathsf{L}^{\nabla}(\rho) \mid$[16]. Considering the set T of variable-free terms of $\mathsf{L}(\rho \cup \mathrm{C})$, form the canonical structure $\mathfrak{H}$, for signature $\rho \cup \mathrm{C}$ as usual. It has universe $\mathrm{H} = \mathrm{T}/\approx$ where $\mathrm{t}' \approx \mathrm{t}''$ iff $\Sigma \vdash^{\mathrm{s}} \mathrm{t}' \simeq \mathrm{t}''$.

We can now provide an upward closed family, by considering the single free-variable formulas of $\mathsf{L}^{\nabla}(\rho \cup \mathrm{C})$, as follows. We consider the set $\varphi^{\Sigma} = \{\mathrm{t}/\approx\in \mathrm{H} : \varphi[\mathrm{v} := \mathrm{t}] \in \Sigma\}$ represented in Σ by formula φ of $\mathsf{L}^{\nabla}(\rho \cup \mathrm{C})$, with single free variable v, form the family ${}_{\nabla}\Sigma = \{\varphi^{\Sigma} \subseteq \mathrm{H} : \nabla \mathrm{v}\varphi \in \Sigma\}$ of those provably sizable, and close it under supersets, to obtain the upward closed family $\mathcal{C}_{\Sigma} = \{\mathrm{X} \subseteq \mathrm{H} : \varphi^{\Sigma} \subseteq \mathrm{X}$, for some $\varphi^{\Sigma} \in_{\nabla} \Sigma\}$.

By our axioms, this upward closed family $\mathcal{C}_{\Sigma} \subseteq \wp(\mathrm{H})$ is proper. So, we use it to expand the canonical structure $\mathfrak{H}$ to an upward closed structure $\mathfrak{H}^{\mathcal{C}_{\Sigma}} = (\mathfrak{H}, \mathcal{C}_{\Sigma})$ for $\mathsf{L}^{\nabla}(\rho \cup \mathrm{C})$.

We can now show, by induction: $\mathfrak{H}^{\mathcal{C}_{\Sigma}} \vDash \tau$ iff $\Sigma \vdash^{\mathrm{s}} \tau$, for each sentence τ of $\mathsf{L}^{\nabla}(\rho \cup \mathrm{C})$. The inductive step for the new quantifier ∇[17], namely, $\mathfrak{H}^{\mathcal{C}_{\Sigma}} \vDash \nabla \mathrm{v}\varphi$ iff $\Sigma \vdash^{\mathrm{s}} \nabla \mathrm{v}\varphi$, follows from the property $\varphi^{\Sigma} \in \mathcal{C}_{\Sigma}$ iff $\varphi^{\Sigma} \in_{\nabla} \Sigma$ (due to schema $[\rightarrow \nabla]$).

We thus have a Löwenheim-Skolem Theorem for our upward closed logic.

Löwenheim-Skolem for upward closed logic

Each consistent set Γ of sentences of $\mathsf{L}^{\nabla}(\rho)$ has an upward closed model $\mathfrak{H}^{\mathcal{C}_{\Sigma}} = (\mathfrak{H}, \mathcal{C}_{\Sigma})$ with cardinality at most $\mid \mathsf{L}^{\nabla}(\rho) \mid$ $(\mid \mathfrak{H} \mid \leq \mid \mathsf{L}^{\nabla}(\rho) \mid)$.

Hence, we have the desired result for our logical system.

[16] The properties of conservative extensions by the addition of witnesses and of maximal consistent extensions for our deductive system can be established as in classical first-order logic (the former by using the Deduction Theorem) by relying on the above connection ($\vdash^{\mathrm{s}}$ & $\vdash$).

[17] The inductive steps for the propositional connectives as well as for the classical quantifiers $\forall$ and $\exists$ are as in Henkin's proof.

Theorem 1: *Completeness of* $\vdash^{\mathrm{s}}$ *with respect to* $\models^{\mathcal{C}}$
The deductive system $\vdash^{\mathrm{s}}$ *is complete with respect to upward closed consequence:* $\Gamma \vdash^{\mathrm{s}} \tau$ *whenever* $\Gamma \models^{\mathcal{C}} \tau$.

4.3. *Other metamathematical properties*

We now examine other metamathematical properties of our logic $\mathcal{L}_{\omega\omega}(\rho)^{\mathrm{s}}$.

We have a sound and complete deductive system for upward closed logic. As usual, such a result transfers the finitary character of derivability $\vdash^{\mathrm{s}}$ to the compactness of the semantical consequence $\models^{\mathcal{C}}$. Thus, our logic is a proper extension of classical first-order logic with compactness and Löwenheim-Skolem properties[18].

Also, our logic $\mathcal{L}_{\omega\omega}(\rho)^{\mathrm{s}}$ for 'several' has some other connections with classical first-order logic $\mathcal{L}_{\omega\omega}(\rho)$: the pleasing fact that it is a conservative extension of classical first-order logic as well as related reductions of some consequences to first-order.

Proposition 2: *Conservativeness of upward closed logic over classical logic*
For each set $\Delta \cup \{\sigma\}$ *of sentences of* $\mathsf{L}(\rho) : \Delta \vdash \sigma$ *iff* $\Delta \vdash^{\mathrm{s}} \sigma$.

Proof outline. A first-order model $\mathfrak{M} \models \Delta$ can be expanded to an upward closed model $\mathfrak{M}^{\mathcal{C}} \models \Delta$, which yields the ($\Leftarrow$) part. The ($\Rightarrow$) part follows the connection ($\vdash^{\mathrm{s}}$ & $\vdash$). □

We will now examine some reductions of generalized consequences of a first-order theory to first-order consequences.

By a *simply generalized formula* we mean one of the form $\nabla \mathrm{v}\varphi$, for some purely first-order formula φ. We will use the idea of *upward closure* of a subset S of the universe V: ${}_{\mathrm{S}}\mathcal{C} = \{\mathrm{X} \subseteq \mathrm{V} : \mathrm{S} \subseteq \mathrm{X}\}$[19].

The next result shows that, in the context of a purely first-order theory, inference and refutation of simply generalized formulas can be reduced to first-order consequences.

[18] The apparent conflict with Lindström's results [13, 3] is explained by our concept of structure (with a family of subsets). Lindström's first theorem asserts: "no *regular* logical system with more expressive power than classical first-order logic has both compactness and Löwenheim-Skolem properties", but the structures for a regular logical system are the usual first-order structures (providing realizations only for the symbols of the signature) [7 (p. 193–199)].

[19] The upward closure ${}_{\mathrm{S}}\mathcal{C}$ is an upward closed family, which is proper iff $\mathrm{S} \neq \emptyset$.

Proposition 3: *Simply generalized sentences and first-order theory*
Consider a set Δ of sentences of $\mathsf{L}(\rho)$ and a formula φ of $\mathsf{L}(\rho)$.

a) *Inference of simply generalized formula $\nabla v\varphi$: $\Delta \vdash^{s} \nabla v\varphi$ iff $\Delta \vdash \forall v\varphi$.*
b) *Refutation of simply generalized formula $\nabla v\varphi$: $\Delta \vdash^{s} \neg\nabla v\varphi$ iff $\Delta \vdash \neg\exists v\varphi$.*

Proof outline. For the($\Rightarrow$) part of each item, we use the upward closure of a nonempty set to expand a first-order model $\mathfrak{M} \vDash \Delta$ to an upward closed model $\mathfrak{M}^{\mathcal{C}} \vDash \Delta$. The ($\Leftarrow$) parts follow from the conservativeness of $\mathcal{L}_{\omega\omega}(\rho)^{s}$ over $\mathcal{L}_{\omega\omega}(\rho)$ and the schemas in 3.3: [$\forall\nabla$], for (a), and [$\nabla\exists$], for (b). $\square$

Example (Theories of solid metals). Consider consistent theories, with a constant h (for mercury) and a unary predicate S (for "being solid"), giving information about which metals are solid under ordinary conditions.

a. First, consider a purely first-order theory Δ, with axioms $\neg S(h)$, $\forall v[\neg v \simeq h \rightarrow S(v)]$ and $\exists v \neg v \simeq h$, expressing "Mercury is not solid", "Every metal, other than mercury, is solid" and "Mercury is not the only metal", respectively.

In this case, we have both $\Delta \nvdash^{s} \nabla v S(v)$ and $\Delta \nvdash^{s} \neg\nabla v S(v)$. So, we cannot decide whether "several metals are solid".

b. Now, consider a consistent theory Γ extending Δ with the generalized information $\nabla v \neg v \simeq h$ for "Several metals are distinct from mercury".

Then, one concludes that "several metals are solid", as $\Gamma \vdash^{s} \nabla v S(v)$. $\square$

We can also reduce to first-order some consequences of an extension of a first-order theory by a single simply generalized axiom. So, the consequences, in this case too, become somewhat trivialized.

Proposition 4: *Extension by a simply generalized axiom and classical logic*
Consider a set Δ of sentences of $\mathsf{L}(\rho)$ and a formula ψ of $\mathsf{L}(\rho)$.

a) *For every formula θ of $\mathsf{L}(\rho)$, we have:*

$$\Delta \cup \{\nabla v\psi\} \vdash^{s} \nabla v\theta \text{ iff } \Delta \vdash \forall v(\psi \rightarrow \theta);$$

$$\text{if } \Delta \cup \{\nabla v\psi\} \vdash^{s} \neg\nabla v\theta \text{ then } \Delta \vdash \neg\exists v(\psi \wedge \theta).$$

b) *For every sentence τ of $\mathsf{L}(\rho)$, we have:*

$$\Delta \cup \{\nabla v\psi\} \vdash^{s} \tau \text{ iff } \Delta \cup \{\exists v\psi\} \vdash \tau.$$

Proof outline. For item (a), we argue much as in the preceding result[20]. Item (b) follows from item (b) of the preceding result by contraposition. □

As a simple example, consider purely first-order information about workers in a plant. Assume that one observes that “several workers are careless”, expressed by $\nabla vC(v)$, and asks whether one can then conclude $\nabla vA(v)$, expressing “several workers are accident prone”. One can infer this generalized assertion iff the first-order information entails the universal assertion $\forall u[C(v) \rightarrow A(v)]$ i.e., “all careless workers are accident prone”.

By examining more closely the expressive power of the generalized quantifier ∇, one can see that the above reductions to classical logic fail for more complex sentences, such as $\exists u \nabla v\ u \simeq v$ [23, 25].

5. *Relative Notions of ‘Several’*

We shall now examine the idea of having a notion of ‘several’ relative to a universe: how it arises and is formulated, as well as some related issues.

We will first indicate how the proper expression of relative ‘several’ assertions brings about the idea of a notion of ‘sizable’ with respect to each universe, leading to its natural formulation in a sorted version of upward closed logic. Then, the need for establishing some connections while blocking others will lead to comparing such relative concepts. Finally, these ideas will be incorporated into a sorted framework for relative ‘several’.

5.1. *The need for relative ‘several’*

Our generalized quantifier ∇ may exhibit somewhat unexpected behavior in some cases. We shall now examine these undesirable side-effects and propose a way to overcome this difficulty.

The generalized quantifier ∇ is meant to capture the idea of holding generally, i.e., for several objects of the universe. Sometimes we wish to express the idea of holding generally over a given subset of the universe, i.e., for several objects of a given sub-universe.

We now examine the expression of such relative several assertions.

On a universe B of birds, we express “Several birds fly” by $\nabla vF(v)$. How are we to express ‘relative several’ assertions, like “Several eagles have wings” or “Penguins generally have beaks”?

[20] For the ($\Rightarrow$) parts, we use the upward closure of a nonempty set to expand a first-order model $\mathfrak{M} \models \Delta$ to an upward closed model $\mathfrak{M}^{\mathcal{C}} \models \Delta$. The ($\Leftarrow$) part follows from conservativeness and the schema $[\rightarrow \nabla]$ in 3.3.

By analogy with the classical quantifiers, relativization is an apparently natural suggestion, i.e., expressing “several M’s are N’s” by $\nabla v[M(v) \rightarrow N(v)]$. Unfortunately, relativization fails to provide an adequate way of expressing ‘relative several’ assertions, due to the behavior of the quantifier ∇.

Example (Penguins and winged birds). Consider expressing the following facts on birds by relativization.

- “All penguins are winged birds” by $\forall v[P(v) \rightarrow W(v)]$.
- “Several winged birds fly” by $\nabla v[W(v) \rightarrow F(v)]$.

From these two sentences, one concludes $\nabla v[P(v) \rightarrow F(v)]$, which would be read as “Several penguins fly” or “Penguins generally fly”.

Now, the two given premises appear to express reasonable facts. On the other hand, the conclusion, *as read*, is not so reasonable[21]. □

This example indicates that relativization fails to express the intended idea. The reason comes from neglecting the relative aspect.

For a generalized formula $\nabla v[M(v) \rightarrow N(v)]$ the reading “several M’s are N’s” is not appropriate. For, one must bear in mind that what this does assert is “for several *birds* b, if M(b) then N(b)”. Indeed, given the (classical) meaning of the conditional, sentence $\nabla v[P(v) \rightarrow \neg F(v)]$ means that the set $P \cap F$ of flying penguins is a small set of *birds* (rather than of *penguins*). Thus, the change in context [17 (p. 166), 2 (p. 217)] is not reflected. This question also appears to be connected to the so-called “Confirmation Paradox” in Philosophy of Science [10][22].

A natural approach to overcome this problem, thereby expressing ‘relative several’ assertions, rests on relative notions of ‘sizable’: each universe has its own relative notion of sizable subsets. This idea may be formulated by giving an upward closed family $\mathcal{C}_V$ over each given universe V.

With relative notions of ‘sizable’, we can express “Several M’s are N’s” as the $\{m \in M : N(m)\}$ is a sizable set of M’s more precisely by $M \cap N \in \mathcal{C}_M$.

[21] One can consistently hold that “Several winged birds fly”, “All penguins are winged birds” and “Several penguins do not fly” (or even “No penguin flies”). Apparently, the set of penguins, being a rather small set of winged birds, does not constitute a sizable set of exceptions to the belief that several winged birds fly.

[22] An example of this paradox is: each flying eagle is considered as evidence in favor of “eagles fly”, whereas a non-flying non-eagle is not, even though “eagles are fliers” and “non-fliers are non-eagles” are logically equivalent. In our case, “several eagles are fliers” and “several non-fliers are non-eagles” appear to involve distinct contexts, namely eagles and non-fliers, respectively.

Thus, we can also distinguish, say, “Several eagles fly” from “Several penguins fly”, since the former becomes $\mathrm{E} \cap \mathrm{F} \in \mathcal{C}_{\mathrm{E}}$, whereas the latter is expressed by $\mathrm{P} \cap \mathrm{F} \in \mathcal{C}_{\mathrm{P}}$.

Having in mind postulate (V) in 2.2, we shall sometimes write $\mathrm{M} \cap \mathrm{N} \in \mathcal{C}_{\mathrm{M}}$ as $\mathrm{M} \cap \mathrm{N} \approx \mathrm{M}$ to suggest the reading “$\mathrm{M} \cap \mathrm{N}$ ‘as sizable as’ M”.

5.2. *Sorted upward closed logic*

A many-sorted approach can provide a framework for formulating the idea of distinct notions of ‘sizable’ for the universes, where one assigns upward closed families corresponding to these relative notions of sizable.

We shall now examine sorted versions of upward closed logic. The basic idea is relativizing to sorts the previous (unsorted) concepts.

We consider many-sorted signatures, where the extra-logical symbols, as well as variables, come classified according to sorts [8]. Quantifiers are relativized to sorts, as expressed in the formation rules:

- for each variable v over sort s, if φ is a formula in $\mathsf{L}^{\nabla}(\rho)$, then so are $(\forall \mathrm{v} : \mathrm{s})\varphi$, $(\exists \mathrm{v} : \mathrm{s})\varphi$ and $(\nabla \mathrm{v} : \mathrm{s})\varphi$.

An *upward closed structure* $\mathfrak{M}^{\mathcal{C}}$ for S-sorted signature ρ is an expansion of an S-sorted first-order structure $\mathfrak{M}$ for ρ, obtained by assigning to each sort s of signature ρ a proper upward closed family $\mathcal{C}_{\mathrm{s}}$ over the universe $\mathfrak{M}[\mathrm{s}]$ of sort s (giving the sizable subsets of $\mathfrak{M}[\mathrm{s}]$).

The extension of *satisfaction* becomes relativized to sorts accordingly:

$(\vDash^{\nabla})_{\mathrm{s}}$ for a generalized formula $(\nabla \mathrm{v} : \mathrm{s})\varphi$, we define $\mathfrak{M}^{\mathcal{C}} \vDash (\nabla \mathrm{v} : \mathrm{s})\varphi[\mathfrak{s}]$ iff the set $\{\mathrm{m} \in \mathfrak{M}[\mathrm{s}] : \mathfrak{M}^{\mathcal{C}} \vDash \varphi[\mathfrak{s}(\mathrm{v} \mapsto \mathrm{m})]\}$ belongs to the family $\mathcal{C}_{\mathrm{s}} \subseteq \wp(\mathfrak{M}[\mathrm{s}])$.

The upward closed *axiom schemas* in the set $\mathrm{B}^{\mathrm{s}}(\rho)$ become sorted as well:

$$
\begin{array}{ll}
[\forall\nabla]_{\mathrm{s}} & : (\forall \mathrm{v} : \mathrm{s})\varphi \to (\nabla \mathrm{v} : \mathrm{s})\varphi, \\
[\to \nabla]_{\mathrm{s}} & : (\forall \mathrm{v} : \mathrm{s})(\psi \to \theta) \to [(\nabla \mathrm{v} : \mathrm{s})\psi \to (\nabla \mathrm{v} : \mathrm{s})\theta)], \\
[\nabla\exists]_{\mathrm{s}} & : (\nabla \mathrm{v} : \mathrm{s})\varphi \to (\exists \mathrm{v} : \mathrm{s})\varphi, \\
[\nabla\nu]_{\mathrm{s}} & : (\nabla \mathrm{v} : \mathrm{s})\varphi \to (\nabla \mathrm{u} : \mathrm{s})\varphi[\mathrm{v} := \mathrm{u}], \text{ for a new u.}
\end{array}
$$

Much as in classical first-order logic, the sorted and unsorted versions are very similar. So, soundness and completeness carry over to the sorted version, by relativizing to sorts the previous arguments. For completeness, the witnesses introduced for the existential quantifiers inherit the corresponding sorts and we thus have sorted families of sizable subsets.

5.3. *Comparing relative notions*

We now take a closer look at the proposal of employing distinct notions of sizable subsets. We shall now examine how the need for establishing some connections while blocking others leads to comparing relative notions of sizable sets. We shall first introduce the ideas and examine some of their features by using variations of the previous examples. In the next section, we shall formulate these ideas precisely in sorted upward closed logic.

The next example shows how some (undesired) conclusions can be blocked.

Example (Birds and penguins with unrelated sizable sets). Given that "All penguins are birds", i.e., $P \subseteq B$, consider the following assertions.

1. "Several birds fly" (the flying birds form a sizable set of *birds*): $B \cap F \in \mathcal{C}_B$.
2. "Several penguins fly" (the flying penguins form a sizable set of *penguins*): $P \cap F \in \mathcal{C}_P$.
3. "Several penguins fail to fly" (the non-flying penguins form a sizable set of *penguins*): $P - F \in \mathcal{C}_P$.
4. "Several birds fail to fly" (the non-flying birds form a sizable set of *birds*): $B - F \in \mathcal{C}_B$.

Now, neither 1 entails 2 (as the non-penguins may form a sizable set of birds) nor does 3 entail 4 (as the *penguins* may very well fail to be a sizable set of *birds*), if the notions of sizable subsets are not related[23]. □

This example illustrates the idea of independent notions of sizable subsets. It is this independence that blocks the undesired conclusions[24].

The next example illustrates how some (desired) conclusions can be achieved.

Example (Birds and winged birds with related sizable sets). Given a universe B of birds, let $W \subseteq B$ be the sub-universe of winged birds. Consider the following assertions.

1. "Several birds fly", as before: $B \cap F \in \mathcal{C}_B$ or $B \cap F \approx B$.
2. "Several winged birds fly", i.e., the flying winged birds form a sizable set of *winged birds*: $W \cap F \in \mathcal{C}_W$ or $W \cap F \approx W$.

[23] We may even have $P \cap F = \emptyset$ and $B \cap F = B - P$; then $B - F = P = P - F$. Now, we may also have $B - P \in \mathcal{C}_B$ (so, 1 holds and 2 does not hold) and $P \notin \mathcal{C}_B$ (so, 3 holds and 4 does not hold), if the notions of sizable subsets are not related.

[24] If few birds are penguins, then one expects the penguins to have little impact on the likelihood of birds flying.

Now, both assertions seem reasonable. In fact, one would expect them to be connected (apparently, due to some strong link between the universes).

To see how these assertions can be connected, we examine some coherent transfer principles, relating notions of sizable subsets.

First, consider the following downward transfer principle

- for every subset $Y \subseteq B$: if $Y \approx B$ $(Y \in \mathcal{C}_B)$, then $W \cap Y \approx W$ $(W \cap Y \in \mathcal{C}_W)$.

In the presence of this downward principle, 2 follows from 1.

Now, consider the following upward transfer principle

- for every subset $Y \subseteq B$: if $W \cap Y \approx W$ $(W \cap Y \in \mathcal{C}_W)$, then $Y \approx B$ $(Y \in \mathcal{C}_B)$.

In the presence of this upward principle, 1 follows from 2. □

This example illustrates the idea of notions of sizable subsets related by coherent transfer principles. Such coherent principles relate families of sizable subsets, thereby enabling relative assertions to be transferred.

A question that remains is when is it reasonable to assume such coherent transfer of sizable subsets. This situation can be clarified as follows.

Given $S \subseteq T$ and an upward closed family $\mathcal{C}_S$ over S, consider the *relativizable family* ${}^{T}\mathcal{C}_S = \{Y \subseteq T : S \cap Y \in \mathcal{C}_S\}$[25]. Then, we can state our transfer principles as follows

upward transfer principle: ${}^{T}\mathcal{C}_S \subseteq \mathcal{C}_T$;

downward transfer principle: $\mathcal{C}_T \subseteq^{T} \mathcal{C}_S$.

Necessary conditions for these inclusions are $S \in \mathcal{C}_T$ and $(T - S) \notin \mathcal{C}_T$, respectively.

Sufficient conditions for adopting such coherent transfer principles fall outside the realm of our logic for 'several'[26]. Some intuitive explanations towards their plausibility can be given by means of sizes and relative frequencies[27]. In the next section, we shall formulate these coherent transfer

[25] Given a proper upward closed family $\mathcal{C}_S$ over a sub-universe $S \subseteq T$, its relativizable family ${}^{T}\mathcal{C}_S$ is an upward closed family $\mathcal{C}_S$ over T, such that $(T - S) \notin {}^{T}\mathcal{C}_S$ and $S \in {}^{T}\mathcal{C}_S$.

[26] This is due to the relative character of sizable as carrying considerable importance, as illustrated in the example in note 6 in 2.1.

[27] For instance, consider the preceding example of birds and winged birds with related sizable sets. The ratio between the likelihoods $| B \cap F | / | B |$, of birds flying, and $| W \cap F | / | W |$, of winged birds flying, will be at least $| W | / | B |$; so $| B \cap F | / | B |$ will be high if both $| W \cap F | / | W |$ and $| W | / | B |$ are high. Also, the difference between the sizes $| B \cap F |$, of the set of flying birds, and $| W \cap F |$, of the set of flying winged birds, will be at most $| B - W |$, so $| W \cap F |$ will be reasonably high if $| B \cap F |$ is high and $| B - W |$ not high.

principles precisely in sorted upward closed logic and examine more closely the impact of their adoption.

5.4. *Sorted framework for relative 'several'*

We shall now consider comparison of universes, with distinct notions of sizable subsets, in a sorted framework. We shall examine how to formulate some ideas related to sub-universes and coherent transfer in this approach.

In our sorted framework, sorts are unrelated: we have equality only over a sort, rather than between distinct sorts. Nevertheless, we can express some relationships among sorts by means of appropriate injections. The idea is that an injection i from s to t establishes a bijection from its domain s onto its image i[s], the latter being a subset of t. Also, the distinction between a set $Z \subseteq t$ and its pre-image $i^{-1}[Z]$ is confined to the non-image $t - i[s]$.

To express that sort s is a subsort of sort t, were sort to a unary function i from s to t together with an axiom asserting its injectivity [14]. This gives transitivity of subsorts. We can now formulate our previous coherent transfer principles under an injection $i : s \to t$ for a subset $Z \subseteq t$ as follows

- *forward transfer principle*: for a subset $Z \subseteq t$, $Z \in \mathcal{C}_t$ whenever $i^{-1}[Z] \in \mathcal{C}_s$,
- *backward transfer principle*: for a subset $Z \subseteq t$, $i^{-1}[Z] \in \mathcal{C}_s$ whenever $Z \in \mathcal{C}_t$.

Necessary conditions for these principles are $i[s] \in \mathcal{C}_t$ and $t - i[s] \notin \mathcal{C}_t$, respectively.

Now, given $i : s \to t$ and formula $\varphi(z)$ with variable z over t, we can express:

- several objects of t are in the image of i (i.e., $i[s] \in \mathcal{C}_t$) by $(\nabla z : t)(\exists x : s)z \simeq i(x)$,
- it is not the case that several objects of t fail to be in the image of i (i.e., $t - i[s] \notin \mathcal{C}_t$) by $\neg(\nabla z : t)\neg(\exists x : s)z \simeq i(x)$,
- several objects of t have property φ by $(\nabla z : t)\varphi(z)$,
- several objects of s map to objects with property φ by $(\nabla x : s)\varphi(i(x))$.

This leads to the following coherent transfer schemas:

- *forward transfer schema* $[s \xrightarrow{i} t : \nabla]$, with the instances

$$
\begin{aligned}
(s \xrightarrow{i} t : \varphi) \quad : \quad & (\nabla z : t)(\exists x : s)z \simeq i(x) \\
& \to [(\nabla x : s)(\varphi[z := i(x)]) \to (\nabla z : t)\varphi];
\end{aligned}
$$

- *backward transfer schema* $[s \xleftarrow{i} t : \nabla]$, with the instances

$$(s \xleftarrow{i} t : \varphi) \quad : \quad \neg(\nabla z : t)\neg(\exists x : s) z \simeq i(x) \\ \rightarrow [(\nabla z : t)\varphi \rightarrow (\nabla x : s)(\varphi[z := i(x)]).$$

Let us now examine versions of our previous examples in this sorted formulation. Recall that we formulate subsorts by means of injections.

Example (Sorted birds and winged birds). Consider sorts b (for birds) and w (for winged birds), as well as a unary predicate F (for flies) over sort b.

Consider the theory Γ, with $j : w \rightarrow b$ and the three axioms

- $(\forall x', x'' : w)[j(x') \simeq j(x'') \rightarrow x' \simeq x'']$ {“all winged birds are birds”};
- $(\nabla z : b)(\exists x : w) z \simeq j(x)$ {“several birds have wings”},
- $\neg(\nabla z : b)\neg(\exists x : w) z \simeq j(x)$ {“it is not the case that several birds are wingless”}.

Consider also the following sentences

- $(\nabla z : b)F(z)$ {“several birds fly”},
- $(\nabla x : w)F(j(x))$ {“several winged birds fly”}.

a. First, in the extension $\Gamma_f = \Gamma \cup [w \xrightarrow{j} b : \nabla]$ of Γ by the addition of the forward transfer schema, the instance $(w \xrightarrow{j} b : F(z))$ yields $(\nabla x : w)F(j(x)) \rightarrow (\nabla z : b)F(z)$. So, since the winged birds form a sizable set of birds, “generally flying” is inherited upwards.

b. In the extension $\Gamma_b = \Gamma \cup [w \xleftarrow{j} b : \nabla]$ of Γ by the addition of the backward transfer schema, the instance $(w \xleftarrow{j} b : F(z))$ yields $(\nabla z : b)F(z) \rightarrow (\nabla x : w)F(j(x))$. So, since the wingless birds are not a sizable set of birds, “generally flying” is inherited downwards.

c. Hence, in the common extension $\Gamma_f \cup \Gamma_b$, we have $(\nabla z : b)F(z) \leftrightarrow (\nabla x : w)F(j(x))$, i.e. the equivalence between “several birds fly” and “several winged birds fly”, showing that “generally flying” is inherited both ways. $\square$

The next example illustrates how the application of the transfer schemas hinges on their antecedents, thereby providing ways to control their impact.

Example (Sorted birds and penguins). Consider sorts b (for birds) and p (for penguins), as well as a unary predicate F (for flies) over sort b.

Consider the theory Δ, with $k : p \to b$ and the following axiom
- $(\forall y', y'' : p)[k(y') \simeq k(y'') \to y' \simeq y'']$ {"all penguins are birds"}.

a. First, given the sentence $(\nabla x : p)\neg F(k(y))$ {"several penguins do not fly"}, form the extension $\Delta' = \Delta \cup \{(\nabla x : p)\neg F(k(y))\} \cup [p \xrightarrow{k} b : \nabla]$ of Δ.

We then have $\Delta' \vdash^s (\nabla z : b)(\exists y : p)z \simeq k(y) \to (\nabla z : b)\neg F(z)$. So, if we assume that the penguins form a sizable set of birds, we will have several birds not flying; but otherwise this conclusion is not forced upon us: $\Delta' \nvdash^s (\nabla z : b)\neg F(z)$.

b. Now, given the sentence $(\nabla z : b)F(z)$ {"several birds fly"}, form the extension $\Delta'' = \Delta \cup \{(\nabla z : b)F(z)\} \cup [p \xleftarrow{k} b : \nabla]$ of Δ.

We then have $\Delta'' \vdash^s \neg(\nabla z : b)\neg(\exists y : p)z \simeq k(y) \to (\nabla x : p)F(k(y))$. So, if we accept that the non-penguins do not form a sizable set of birds, we will have several penguins flying, but otherwise we do not have to accept this conclusion: $\Delta'' \nvdash^s (\nabla x : p)F(k(y))$. $\square$

These examples illustrate how the coherent transfer schemas can provide uniform control based on the relative importances of the sorts.

In the sorted framework for 'several', we consider sorted theories specified by a union $\Omega = \Gamma \cup \Sigma \cup \Delta$ of sets of axioms, where Σ codes (basically syntactical) subsort information, Δ gives coherent transfers between some subsorts[28], and Γ expresses the remaining available knowledge. We thus have $\Omega \vdash^s \tau$ iff $\Omega \vDash^{\mathcal{C}} \tau$.

6. *Conclusions*

We have examined logical systems with generalized quantifiers over upward closed families, which provide rigorous bases for qualitative reasoning with vague notions often rendered as 'several' or 'many'. The unsorted logical system is a conservative extension of classical first-order logic, with which it shares various properties. More interesting situations, however, require assertions relative to several universes, leading to the ideas of 'relative several' and to our sorted framework for them.

[28] As mentioned before, such decisions involve extra-logical considerations and are outside the realm of our logic for 'several'.

We can similarly introduce generalized quantifiers for the dual notion of 'negligible'. Modal versions of these logics can also be contemplated[29].

These logical systems, though related to default logics, are quite different, both technically and in terms of intended interpretations [18]. Our upward closed logic belongs to a family of closely related systems with generalized quantifiers for qualitative reasoning about vague notions [9, 4], including one with filters, for 'most', and one with ultrafilters [22, 23][30].

These systems, which are undergoing further investigation [25], appear to have some interesting connections with fuzzy logic [26][31] as used in expert systems [21], natural language [2, 15] and empirical reasoning [10]. Such connections suggest the possibility of other applications [5, 22, 25].

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[29] We may consider modal versions of these generalized quantifiers for expressing that a formula holds at several worlds (e.g., at several instants) or that a formula holds at several worlds among those accessible from the present world.

[30] These three logics form an increasing chain. Filter logic is appropriate, for instance, for dealing with a notion of 'typical' [25]. The formulation of the coherent transfer principles in section 5 can be considerably simplified in filter and ultrafilter logics [5, 22]. There are also other related versions of these logics, conceived with the aim of capturing similar notions [4, 9].

[31] For instance, we may express a fuzzy notion like 'quite tall' by "taller than several people".

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