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# ON CAUSATION AND A COUNTERFACTUAL IN QUANTUM LOGIC: THE SASAKI HOOK

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*Abstract*
We analyze G.M. Hardegree's interpretation of the Sasaki hook as a Stalnaker conditional and explain how he makes use of the basic conceptual machinery of OQL, i.e. the operational quantum logic which originated with the Geneva Approach to the foundations of physics. In particular we focus on measurements which are ideal and of the first kind, since these encode the content of the so-called Sasaki projections within the Geneva Approach. The Sasaki projections play a fundamental role when analyzing the condition under which the properties expressed by Sasaki hooks can be considered as actual. We finish with a note on how the Sasaki hook can be conceived as "assigning causes for properties to be actual", which links the interpretation of G.M. Hardegree to what has been called "dynamic OQL".

## 1. *Introduction*

Research in the field of quantum logic has pointed out that there are certain problems with an implication connective. Under the condition that an implication should satisfy the strengthened law of entailment, i.e. $a \rightarrow b = 1$ if and only if $a \leq b$, there are in the case of a non-boolean orthomodular lattice five possible connectives one can introduce (see [20]). All five connectives are internally definable as specific two-variable lattice polynomials. One of these polynomials is referred to as the Sasaki hook, i.e. $a \xrightarrow{S} b = a' \vee (a \wedge b)$. Viewed as a connective, the Sasaki hook satisfies the strengthened law of entailment but still behaves quite badly in the non-boolean orthomodular case, due to problems with a deduction theorem (see [21]). In [11] we analyze the Sasaki hook in detail and conclude that one should not conceive it as a

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static implicative connective, though as inducing a labeled dynamic implication that assigns causes within the context of dynamic operational quantum logic (DOQL). Against this background, we will in this paper analyze G.M. Hardegree's interpretation of the Sasaki hook as a Stalnaker conditional and argue that:

> The counterfactual nature of the Sasaki hook, viewed from an operational perspective, shows explicitly that $(a \stackrel{S}{\rightarrow} b)$ is the weakest property which, if it is actual in some state $p$, guarantees that $b$ will be an actual property of the system under the condition that we obtain a positive response from performing the ideal first kind measurement associated to property $a$. In other words, our analysis supports the claim made in [11] that the action of the Sasaki hook assigns causes and has a fundamental dynamic nature. Therefore, the Sasaki hook should indeed not be conceived as a static implication.

We start in section 2 with a brief overview of the theory called Operational Quantum Logic (OQL). In particular, we introduce the concept of a property lattice $\mathcal{L}$ which is complete, atomic, orthomodular and satisfies the covering law. In section 3 we explore the area of the ideal measurements of the first kind from within the OQL framework while following [27] and the recent analysis put forward in [31]. In particular we focus on how an ideal first kind measurement of a property $a \in \mathcal{L}$ can encode the action corresponding to a Sasaki projection represented as the map $\varphi_a^* - : \mathcal{L} \rightarrow \mathcal{L} : b \mapsto a \wedge (a' \vee b)$. In case we deal with a quantum system, it are exactly the ideal first kind measurements with a positive result, which allow us to calculate the perturbation suffered by the system in course of the associated action. Let us remark that an ideal first kind measurement $\alpha$ of property $a$ yielding a positive result, corresponds in Hilbert Space terms to the action of the orthogonal projector $P_a$ on the subspace $a$ and as such exactly encodes a Sasaki projection. In section 4 we concentrate on G.M. Hardegree's interpretation of the Sasaki hook as a Stalnaker conditional in [14, 15], which leads to the statement that $(a \stackrel{S}{\rightarrow} b)$ is actual in at least those states $p \in \Sigma$ which are such that they can actually be transformed in a state in which $a$ is actual and $b$ is actual. Otherwise worded, the states in which $(a \stackrel{S}{\rightarrow} b)$ is actual are at least those in which $b$ is actual after a positive response is obtained in case the system would be submitted to an ideal first kind measurement of property $a$. After concentrating on the problem of inaccessible states, we can state that $(a \stackrel{S}{\rightarrow} b)$ is actual in a state $p$ if and only if either the state in which $a$ is actual is inaccessible for $p$ or $b$ is actual in the state $\varphi_a^*(p)$ in which $a$ is forced to be actual. This leads us to the interpretation of $(a \stackrel{S}{\rightarrow} b)$ as assigning causes, on which we briefly comment in the final section. The causal assigning nature of the Sasaki hook has to be seen against the background of "the Sasaki adjunction as expressing a specific causal duality" [11].

## 2. *Operational Quantum Logic*

The idea of a logical calculus based on the relation between the properties of a physical system and the projections defined on a Hilbert space, can be traced back to J. von Neumann's [33], though a first profound analysis, in which one focuses on experimental propositions instead of properties, appeared in [5]. The latter is seen as the onset of quantum logic in which G. Birkhoff and J. von Neumann analyzed why the logic underlying the formalism of quantum theory is not classical. The main point being that the structure of the mathematical representatives for experimental propositions of a quantum system, corresponding to the projections on a Hilbert space (or isomorphically, to the closed subspaces), since the failure of distributivity forms an orthomodular lattice. We come back to the structure of an orthomodular lattice in detail below. Let us for now just mention that such an orthomodular lattice is an orthocomplemented lattice satisfying weak modularity, i.e. $a, b \in \mathcal{L}$, $a \leq b$ implies that $b = a \vee (b \wedge a')$, and not the by G. Birkhoff and J. von Neumann proposed stronger modular law. Besides the difference between classical and quantum structures we like to draw attention to another important point. As mentioned also in [12], the structure of "all" closed subspaces in a Hilbert space, closed under the logical conjunction, amounts to a complete orthomodular lattice and this has been a point of struggle for many quantum logicians. Indeed, considering a logical conjunction for every set of experimental propositions, has often been thought of as too strong a requirement since its physical meaning would be underdefined in some cases. This is based on the idea that, especially for quantum systems, not any two experimental propositions can really be tested at the same time. Because of such difficulties the quantum logic community abstracted more and more from Birkhoff and von Neumann's approach. We however believe that completeness should not be given up and follow the Geneva Approach in which one actually does stick close to Birkhoff and von Neumann's ideas. Roughly following the classifications in [17, 22] of different areas of work in quantum logic, the Geneva Approach is a so-called (physical) operational approach. Operational Quantum Logic (OQL), used here as another name for the Geneva Approach to the foundations of physics, originated with the work done in [1, 18, 19, 25, 27, 30] and corresponds to the theory of "Property Lattices". The underlying motive is to give a complete formal description of a physical system in terms of its actual and potential properties and a dual description in terms of its states. In the next paragraphs we give a brief introduction to how this can be achieved, following the recent analysis of OQL in [7, 8, 23, 31].

The first notion we introduce is that of a "(yes-no) question" or "definite experimental project" as it more recently has been called in [23]. A yes-no question $\alpha \in Q$ is an experimental procedure and can be thought of as a list of concrete actions which is accompanied by a rule that specifies in advance which outcome(s) count as a positive response. Of all the outcomes which do not yield the positive response "yes", we say that they yield the negative response "no". A question is called "true" for a particular physical system if it is sure that "yes" would obtain should we perform the experimental procedure, and is called "not true" otherwise. In case it is sure that "no" would obtain, then we also call the question "impossible". There is a "trivial question" $I \in Q$ which always will yield a positive response no matter what you do with the system while the "absurd question" $O \in Q$ always will yield a negative response. The product question $\Pi_J \alpha_i$ consists of arbitrarily choosing one question from a given family of questions $\alpha_i$ and performing it. The answer obtained from effectively performing the arbitrarily chosen question is the answer attributed to the product question. For a particular physical system, $\Pi_J \alpha_i$ is true if and only if each $\alpha \in \alpha_i$ is true. Finally we obtain the inverse question $\alpha^\sim$ of a particular question $\alpha$ by exchanging the responses "yes" and "no". The collection of possible questions which could be performed on a particular physical system is preordered in a natural way by means of the relation $\prec \subseteq Q \times Q$ which expresses that every time when $\alpha$ is true, $\beta$ is true as well, i.e. $\alpha \prec \beta$. Here $I$ is seen as the maximal element and $O$ as the minimal element.

Next we consider the properties of a physical system, written as $a, b, ... \in \mathcal{L}$, construed as candidate elements of reality corresponding to the yes-no questions defined for a particular physical system. We say that there is a one-to-one correspondence between: 1) the property $a \in \mathcal{L}$, associated with the question $\alpha$ and 2) the equivalence class of questions $[\alpha]$ which is the collection of all questions $\beta$ for which $\alpha \prec \beta$ and $\beta \prec \alpha$. In other words, to each equivalence class of questions there "corresponds" a property and we use the notation $\zeta(\alpha) = a$ to express that $a$ is associated with $\alpha \in [\alpha]$. We call a property of a particular physical system "actual" if and only if the questions which test it are true and potential otherwise. The link between the actuality of properties and the truth of questions allows us to state that the preorder relation on questions induces a partial order relation on the set of all properties. In particular we write $a \leq b$ if and only if $\alpha \prec \beta$ with $\zeta(\alpha) = a$ and $\zeta(\beta) = b$. Within OQL, one proves now that the set of all properties of a physical system is indeed a complete lattice (see for instance [30]). It is the product question which operationally ensures the existence of a greatest lower bound (meet) for any subset of properties: $\wedge_J a_i = \zeta(\Pi_J \alpha_i)$ with $\zeta(\alpha_i) = a_i$, yielding as such a complete meet semilattice. Birkhoff's theorem then ensures that we also have the least upper bound of every subset of $\mathcal{L}$ in the following way: $\vee_J a_i = \bigwedge\{b \in \mathcal{L} \mid \forall i \in J : a_i \leq b\}$. Another important notion is that of the state of a particular physical system. States $\mathcal{E}_1, \mathcal{E}_2, ... \in \Sigma$ are construed as abstract names which encode the possible singular realizations of the system. Within OQL, the actuality or potentiality of a property will depend on the state of the system. As such the state stands in a correspondence relation to the system's actual properties, which allows us to introduce the map $S : \Sigma \to P(\mathcal{L}) : \mathcal{E} \mapsto S(\mathcal{E}) = \{a \in \mathcal{L} \mid a\ is\ actual\ in\ \mathcal{E}\}$. Under the assumption that $\mathcal{L}$ is atomic, we represent states by atoms of $\mathcal{L}$, i.e. a state $\mathcal{E}$ can be represented by $p_{\mathcal{E}} = \wedge_{a \in S(\mathcal{E})} a$ which is the strongest property in $\mathcal{L}$ that is actual in the state $\mathcal{E}$. We are then allowed to call $\mathcal{L}$ atomistic, meaning that every $a \in \mathcal{L}$ can be written as $\bigvee\{p \leq a \mid p\ is\ an\ atom\}$ (see theorem 1 in [23]). On the set of all possible states of a particular physical system we introduce an orthogonality relation $\perp \subseteq \Sigma \times \Sigma$, which is symmetric and antireflexive as follows: $\mathcal{E}_1 \perp \mathcal{E}_2$ if and only if there is an $\alpha$ which is true for $\mathcal{E}_1$ and impossible for $\mathcal{E}_2$. Note that $\alpha$ is true for $\mathcal{E}_1$ in case $a = \zeta(\alpha) \in S(\mathcal{E})$. Within OQL one then introduces the axiom stating that for every given state $\mathcal{E}$ there exists at least one question which is true in a possible state if and only if that state is orthogonal to $\mathcal{E}$. This axiom and the link between states and properties allows us formally to explain that properties have "opposite" properties. And when we postulate that each property is the opposite of another one, we equip $\mathcal{L}$ with an orthocomplementation, i.e. a surjective map $' : \mathcal{L} \to \mathcal{L}$ for which $a \leq b \Rightarrow b' \leq a'$, $a \wedge a' = 0$, $a \vee a' = 1$ and $a'' = a$. Coming back to the above announced state-property dual description of a physical system, we indeed see such a parallelism once we associate with each property $a$ the set of states $\mu(a)$ in which it is actual and to each state the set $S(\mathcal{E})$ of its actual properties. Note that $\mu : \mathcal{L} \to P(\Sigma) : a \mapsto \mu(a) = \{\mathcal{E} \in \Sigma \mid a \in S(\mathcal{E})\}$. If a system satisfies the mentioned axioms, the application of $\mu$, i.e. the so called Cartan map, determines an isomorphism between $\mathcal{L}$ and $(\Sigma, \perp)$ which is the lattice of biorthogonally closed subsets of $\Sigma$ [30].

Within OQL we can now describe both classical and quantum systems. A system is considered as classical if for every property $a \in \mathcal{L}$, there is at least one $\alpha$ such that $a = \zeta(\alpha)$ and which gives a result (yes or no) that is certain *a priori* from the moment on that the system is realized in a specific state [30]. For a classical system a yes-no question or its inverse is true in every possible state of the system while for a pure quantum system this is only the case for $I$ and $O$. For the latter kind, C. Piron proved in [25] that a property lattice which is complete, atomic, orthocomplemented and additionally weak modular, which satisfies the covering law and some additional properties —

i.e. the lattice should also be irreducible and of a sufficient dimension — can be represented in the form of the lattice of closed linear subspaces of a generalized Hilbert space. There is a quantum axiom involved here which is the following, (see for instance [2, 27]),

> Suppose $\mathcal{L}$ is a complete and orthocomplemented lattice, then:
> (1) $\mathcal{L}$ is weak modular iff for $a, b \in \mathcal{L}$ and $a \leq b$ we have $b = a \vee (b \wedge a')$
> (2) $\mathcal{L}$ satisfies the covering law iff for $\mathcal{L}$ atomic (which means that for every $a \in \mathcal{L}$ there exists an atom $p$ of $\mathcal{L}$ such that $p \leq a$) and $a, x \in \mathcal{L}$ and $p$ is an atom of $\mathcal{L}$ whenever we have $a \wedge p = 0$ and $a \leq x \leq (a \vee p)$ then holds: $x = a$ or $x = a \vee p$.

This *quantum axiom* as we call it, has a very mathematical nature according to D. Aerts in [1] and captures the reason why it is problematic to describe separated systems by means of a quantum theory. Notice that the property lattice which is weak modular and orthocomplemented has been called *orthomodular*.

## 3. *Ideal Measurements of the First Kind*

Before we dealt with properties being actual or potential regardless of whether or not the associated questions — definite experimental projects — were actually going to be performed. In this section we will introduce some special kinds of questions which allow us to deal with properties — being actual or potential — after these questions have been performed in reality. When we intend to perform an arbitrary question in reality, we can *in general* only say of the properties that the system had before we performed the question, that a negative response *proves* the tested property was not actual beforehand, while a positive response does not prove anything. In the latter case the property may or may not have been actual before the question was performed. It is also a fact that when we do perform questions, we may disturb or even destroy the given physical system. For this reason we concentrate on the notion of a *measurement of the first kind* as introduced by W. Pauli in [24]:

> “The method of measurement ... has the property that a repetition of measurement gives the same value for the quantity measured as in the first measurement. In other words, if the *result* of using the measuring apparatus is not known, but only the fact of its use is known ..., the probability that the quantity measured has a certain value is the same, both before and

> after the measurement. We shall call such measurements the *measurements of the first kind*." (p. 75)

Clearly such a measurement does not destroy the system and if one carries out the measurement a second time, one obtains the same result [27]. Following now the definition of a first kind measurement as given in [27] (our insertion of brackets):

> "A question $\beta$ is called a measurement of the first kind if, every time the answer is "yes", one can state that the proposition [property] $b$ defined by $\beta$ is true [actual] immediately after the measurement." (p. 68)

In other words, a question $\beta$ is a measurement of the first kind if "yes" implies the actuality of the property $b = \zeta(\beta)$ associated with the measurement. We call this question $\beta$ true and $b$ actual if $b$ remains actual and $\beta$ remains true should we repeatedly perform the associated experimental procedure. Our analysis shows that in case we perform such a measurement, and the positive response obtains, then this is a measurement which;

> 1) did not destroy or severely damage the physical system, and
> 2) did not alter the resulting property in case this property was already actual before we performed the experiment

Here 1) states that in order to be able to obtain an actual property afterwards, the system should be somehow preserved, while 2) expresses that if the question $\beta$ is a measurement of the first kind and if $\beta$ is true before we perform the experimental project, obviously the answer "yes" will obtain when we perform the experiment. Since in this case, $b = \zeta(\beta)$ was actual before the performance of the experiment, and if the answer "yes" obtains, it is still actual immediately afterwards while the question $\beta$ is true immediately afterwards.

It should be noted that only when we obtain a positive response for such a measurement, it makes sense to characterize it as first kind. When we perform such a measurement which gives a negative response, there are two possible scenario's. First we are dealing with a situation in which we severely damaged the physical system, this can lead to the case of a so called filter.[1] Second, we are dealing with a situation in which repeating the measurement will not with certainty give the same result, i.e. under the assumption

[1] Filters are experiments which preserve the system every time a certain response is obtained, but which destroy the system otherwise, see [27].

that this measurement is not impossible in a stable manner after its performances since that could lead us to the case of a perfect measurement.[2] The first scenario gave rise in [31] to the statement that a measurement $\beta$, of the first kind, is only a non-demolition experiment with certainty in the case of a positive response. The second scenario points out that a non-demolition measurement with a negative response gives us some information about the associated property before the measurement was performed. This being the case, this situation can been linked to the notion of *a measurement of the second kind*, following W. Pauli in [24]:

> "On the other hand it can also happen that the system is changed but in a controllable fashion by the measurement — even when, in the state before the measurement, the quantity measured had with certainty a definite value. In this method, the result of a repeated measurement is not the same as that of the first measurement. But still it may be that from the result of this measurement, an unambiguous conclusion can be drawn regarding the quantity being measured for the concerned system before the measurement. Such measurements, we call the *measurements of the second kind*." (p. 75)

Measurements of the first kind with a positive response, cannot guarantee that the system has not altered during the course of action. Some potential properties might have become actual and vice versa. So we wish to focus on measurements which are such that they perturb the system as little as possible. In this sense we will consider measurements which leave actual properties as much as possible intact when they are compatible[3] with the property associated with the measurement. This means immediately that measurements which are associated with compatible properties but which cannot be performed together without severely disturbing each other's result — for an example of this kind of situation we refer to [1, p. 39] — will not count as the candidate measurements we have in mind. Still it will be possible that a system is changed when the properties that are actual in the initial state of the system are not compatible with the property associated with the experiment we want to perform (see also [28]). Besides this natural cause of a system's change due to incompatibilities, there is still another possible

[2] $\alpha$ is called a perfect measurement if both $\alpha$ and $\alpha^{\sim}$ are ideal measurements of the first kind [27] — see below for the introduction of the notion of an "ideal first kind measurement".

[3] We work with the notion of "compatibility" in the sense of C. Piron. This means that for $A = \{a, b\} \subset \mathcal{L}$, where $\mathcal{L}$ is at least an orthocomplemented property lattice, we call $a$ and $b$ compatible if the sub-ortholattice generated by $A$ is distributive.

cause. A system can also be perturbed due to imperfections of the measuring apparatus. One could indeed calculate the exact influence of a particular measuring apparatus on the physical system, but in order to allow a more general description independent of any particular apparatus, we need to idealize the situation so that the system is indeed as little perturbed as possible. In an ideal case we presume that the apparatus itself does not have those imperfections which lead to perturbations and so we *demand* that *all* initial actual properties compatible with the property associated with the measurement remain actual after a positive response is obtained. Note that in real life such an ideal case is only reachable by approximation. Let us now introduce the formal definition of an *ideal measurement*, found in [27] (our insertion of brackets):

> "A question $\beta$ is said to be ideal if every proposition [property], compatible with the proposition [property] $b$ defined by $\beta$, which is true [actual] beforehand is again true [actual] afterwards when the response of the system is "yes"." (p. 68)

If we perform an ideal measurement $\beta$ and obtain a positive response, then we are dealing with a measurement which;

> 1) did not destroy or severely damage the physical system
> 2) did not alter $\zeta(\beta) = b$ nor its compatible actual properties in case $\beta$ was true before we performed it.
> 3) did not alter the actual properties, compatible with $\zeta(\beta) = b$, in case $\beta$ was not true before we performed it.

Consider the case in which we perform an ideal measurement $\beta$ and we do not obtain a positive response. This implies that we either destroyed or severely damaged the physical system; or that we did not but that $\beta$ cannot have been true beforehand.
Let us recapitulate: if $\beta$ is ideal and the result is "yes", we know that the state afterwards consists of actual properties among which we find those properties which are compatible with $\zeta(\beta) = b$ and that were actual in the state before we performed $\beta$. In other words, all questions attached to the system which, when performed together with that ideal measurement, were true beforehand, remain true afterwards. If on top of this, the property associated with the ideal measurement was already actual before, we know that the initial state is not perturbed at all by the performance of that measurement [27]. Note that by means of the compatibility-claim of an ideal measurement we indeed maximally limited the perturbation of the system as much as it lies in our power to do so in case of a positive response.

We will now combine both the above given definitions in the following sense,

> An ideal measurement of the first kind of a property $a$ is a question $\alpha$ which satisfies the following three conditions [23, 30]:
> (i) $a = \zeta(\alpha)$ and $a' = \zeta(\alpha^{\sim})$
> (ii) if the positive response is obtained then $a$ is actual immediately after the measurement;
> (iii) if the positive response is obtained then the perturbation suffered by the system is minimal.

In the case of a classical system, an ideal measurement of the first kind is a classical question which does not change the system, at least when the response is positive [30]. In the case of a quantum system, in general, the ideal measurement of the first kind perturbs the system [29]. This is due to the possibility of incompatible properties. Now, *in general* if a property which is being measured is not compatible with every property actual in the initial state, then the result of the measurement is impossible to predict exactly, but when we deal with an ideal first kind measurement the final state is determined immediately after the measurement if the response is "yes" [26]. More precisely, in the case we deal with a non-classical system, an ideal measurement of the first kind with a positive response, allows us to calculate the perturbation suffered by the system in the course of action [27]. This is exactly what the following theorem, proved in [27], is about (our insertion of brackets):

> "If a question $\beta$ is an ideal measurement of the first kind and if the answer is "yes", then the state of the system immediately after the experiment is [represented by] $(p \vee b') \wedge b$, where $p$ is [represents] the state before, and $b$ is the proposition [property] defined by $\beta$." (p. 68)

If we suppose that $b$ — the property associated with the ideal and first kind measurement $\beta$ — was already actual in the initial state, then the initial state will not have been changed by performing $\beta$ because: $b \wedge (p \vee b') = p$ iff $p \leq b$ (see [27, p. 69]). Obviously, when $p$ represents the initial state, and $p \leq b$ then $p \leq (p \vee b) \wedge (p \vee b')$ implies $p \leq b \wedge (p \vee b')$, and because $b \wedge (p \vee b')$ is an atom (due to the covering law) it is equal to $p$.

In case $b'$ was actual before we perform an ideal, first kind measurement $\beta$, we know that $\beta$ cannot give a positive response, because if it did, $b = \zeta(\beta)$ as well as $b'$ would be actual afterwards which is factual impossible. Remark that for an orthomodular lattice $\mathcal{L}$ an arbitrary property and its orthocomplement are necessarily compatible. Henceforth this comes down to the statement $(p \vee b') \wedge b = 0$ iff $p \leq b'$, proved in [27, p. 69, 70].

The change of state which takes place due to an ideal measurement of the first kind $\beta$ with a positive response, can be seen as the map $\varphi_b^*$ from the initial state into the final state: $p \mapsto (p \vee b') \wedge b$. Analyzing the final state we see immediately that $b = \zeta(\beta)$ is actual *and* that the initial state has been weakened since $p \vee b'$ is actual afterwards. This weakening of the initial state is due to the (smallest possible) perturbation of the system and counts as a loss of information. With Piron's work in mind, take for instance the lattice of closed subspaces $\mathcal{L}(H)$ of a Hilbert space, ordered by inclusion, as an example of a pure quantum property lattice. We can now say with [27] that it are exactly the projectors such as $P_b$, which orthogonally project the vectors of $H$ onto the subspace $b$, which induce the maps such as $\varphi_b^*$ for quantum systems. Indeed, in [27] the map $\varphi_b^*$ is proven to have all the formal properties of a projector. Furthermore, $\varphi_b^*(p) = (p \vee b') \wedge b$ is a *Sasaki projection*.

In the next section we use this link between the Sasaki projection and ideal first kind measurements to shed more light on the operational interpretation of the Sasaki hook as a Stalnaker conditional initiated by G.M. Hardegree.

## 4. *The Counterfactual Nature of the Sasaki Hook*

As mentioned in the introduction, given an orthomodular lattice, the Sasaki hook as a connective satisfies the strengthened law of entailment, but still in general does not come close enough to what one from an "implication" would expect. Besides the problems with a deduction theorem, in [14, 15] it is shown that the Sasaki hook — similar as the counterfactual or subjunctive conditional — in the non-boolean orthomodular case falls short of certain laws namely strong transitivity $(a \overset{S}{\to} b) \wedge (b \overset{S}{\to} c) \leq (a \overset{S}{\to} c)$, weakening $(b \overset{S}{\to} c) \leq ((a \wedge b) \overset{S}{\to} c)$ and contraposition $(a \overset{S}{\to} b) = (b' \overset{S}{\to} a')$.[4] As such Hardegree tried to give an interpretation of the Sasaki hook in terms of a Stalnaker conditional. In a way this is quite interesting for us since he links this interpretation to OQL, [15]:

> "I now wish to argue that this interpretation is not a mere formal exercise, but is in fact intimately related to the usual

[4] The failure in strong transitivity and contraposition of the Sasaki hook is pointed out explicitly by means of an example of a non-boolean orthomodular lattice in [16]. Due to the fact that weakening $(b \overset{S}{\to} c) \leq ((a \wedge b) \overset{S}{\to} c)$ can be obtained from strong transitivity $(a \overset{S}{\to} b) \wedge (b \overset{S}{\to} c) \leq (a \overset{S}{\to} c)$ by exchanging the $a$ for $a \wedge b$, in Herman, Marsden and Piziak's example also weakening fails for the Sasaki hook.

> operational characterization of quantum propositions in terms of binary (yes-no) experiments." (p. 72)

His main point in [14] is built up by means of his notion of a filter, where an $a$-filter is to be interpreted as a yes-no question which gives a positive result in case the system possesses the property $a$ or, as Hardegree puts it, in case the system passes the $a$-filter. These filters are used to express the conditional property $c(a, b)$ as follows:

> "If $s$ *were* to pass the $a$-filter, *would* it then (with certainty) pass the $b$-filter? A system $s$ in state $\mathcal{E}$ for which the answer to this question is affirmative is said to have property $c(a, b)$, ..." (p. 416)[5]

As such he can now give a counterfactual interpretation to $(a \xrightarrow{S} b)$ by means of the conditional property $c(a, b)$, [14]:

> "Thus if (and only if) a system $s$ is in state $\mathcal{E}$, which satisfies $(a \xrightarrow{S} b)$, then $s$ has property $c(a, b)$ to the effect that if $s$ were to pass an $a$-filter, then it would also pass a $b$-filter. Whether a given system has this property usually depends on its state;..." (p. 416)

And we go further with, [15]:

> "In certain well specified circumstances, however, $s$ will have property $c(a, b)$ regardless of its state. This is to say that no matter what initial state $s$ is prepared in, if $s$ passes an $a$-filter, it is certain to pass any immediately subsequent $b$-filter. This corresponds to the customary operational characterization of the *relation* of implication among binary experiments pertaining to system $s$; ..." (p. 74)

We will investigate exactly what these quotes mean in the context of OQL. Remark first that when we focus on state transitions we have to take into account that one state can only be transformed into another one when the latter is accessible (not orthogonal) to it. However if we look at $(a \xrightarrow{S} b)$ we can run into problems with such an accessibility criterion. Indeed, $(a \xrightarrow{S} b)$ is valid in at least those states $\mathcal{E}_i$ which are such that they can actually be transformed in a state in which $a$ is actual and $b$ is actual. Otherwise worded, the states in which $(a \xrightarrow{S} b)$ is actual are at least those in which $b$ is actual after a positive response is obtained in case the system would be

[5] Note that we slightly changed Hardegree's notations for properties so that it fits with our symbolism.

submitted to an ideal first kind measurement $\alpha$ with $\zeta(\alpha) = a$. Our idea of linking Hardegree's interpretation to the above explained notion of an ideal first kind measurement, came from the use of the words "minimally different" by Hardegree in the following, [15]:[6]

> "The basic intuition is fairly clear: in deciding the truth of a counterfactual conditional $(A > B)$ in a given situation $x$, we consider (envisage) a particular situation $y$ which makes $A$ true and which is in some relevant sense *minimally different* from $x$. The conditional $(A > B)$ is then true in situation $x$ exactly if $B$ is true in situation $y$." (p. 68)[7]

Against the background of the strengthened law of entailment, we see that $(a \stackrel{S}{\rightarrow} b)$ is valid, i.e. always actual, in every possible state of the system when $a \leq b$. In other words, when $a \leq b$ then $\mu(a \stackrel{S}{\rightarrow} b) = \Sigma$ and this is exactly the case Hardegree refers to when he says that "in certain circumstances, $s$ will have property $c(a, b)$ regardless of its state". This however leads to a small problem when, given $p_{\mathcal{E}} \leq a \leq b$, $\mathcal{E}$ is not accessible to every other state of the system. Unless we solve this problem, it renders the given operational interpretation of $(a \stackrel{S}{\rightarrow} b)$ inadequate. Of course the problem of inaccessible states can also occur when $a \not\leq b$. Note that in the latter case it can still be so that $(a \stackrel{S}{\rightarrow} b)$ is actual in certain states though in general this will not be so in all possible states of a system. Let us now concentrate on solving the problem of inaccessible states. Take the example of $(a \stackrel{S}{\rightarrow} b)$ in a complete orthomodular lattice, the system will not give a positive response to an ideal first kind measurement of $\alpha$ with $\zeta(\alpha) = a$ if $a'$ is actual in the initial state and, as we saw above, the state afterwards will then formally be $(p \vee a') \wedge a = 0$. But since it can well be that $\mu(a') \subseteq \mu(a \stackrel{S}{\rightarrow} b)$, Hardegree defines for the consistency of his counterfactual interpretation of the Sasaki hook an absurd world (state) 0. He then treats the absurd world as the nearest world in which sentence $A$ is true if and only if there are no worlds accessible in which sentence $A$ is true. Following his definition in [14]:

[6] For the role played by yes-no experiments in the interpretation of the Sasaki hook as a counterfactual, see also [4].

[7] Note that Hardegree works here with the algebra of subsets of a state space, where $x$ and $y$ denote states or worlds and $A$ and $B$ denote elementary sentences. An elementary sentence has the form "magnitude ... has value in (Borel set) ...". $(A > B)$ denotes the so-called Stalnaker conditional which is: "the sentence — provided it exists — which is true at any state $x$ exactly if $B$ is true at the nearest $A$-state to $x$." [15, p. 68].

> “We also define the *absurd world* 0 to be the world which satisfies every sentence, including contradictions.” (p. 413)

Indeed if we see 0 as an absurd state in Hardegree’s sense, we have found a way to avoid the problem which may arise due to inaccessible states. And as a consequence we can say that for a complete orthomodular lattice: if $a'$ is actual in the initial state $p$, then $(a \xrightarrow{S} -)$ is actual in $p$, or more generally, [15]:

> “As an immediate consequence of these considerations, if a sentence $A$ is impossible relative to $x$, then $x$ satisfies every conditional of the form $(A > B)$. A *fortiori*, if $A$ is contravalid (true nowhere except 0), then every conditional of the form $(A > B)$ is valid (true everywhere).” (p. 69)

Furthermore, Hardegree gave the formal analysis in the lattice of closed subspaces of a Hilbert space of what it means for $(a \xrightarrow{S} b)$ to be valid in a state $\mathcal{E}$ [14, 15]. While we will avoid the technicalities, we follow [12] in the interpretation of $\mathcal{E} \models (a \xrightarrow{S} b)$:

> $(a \xrightarrow{S} b)$ is actual (true) in a state $\mathcal{E}$ iff either $a$ is impossible for $\mathcal{E}$ or $b$ is actual in the state into which $\mathcal{E}$ can been transformed after performing a positive $a$-question, and which verifies $a$.

Where the notion of “$a$ is impossible for $\mathcal{E}$” may here be interpreted as: the state in which $a$ is actual is inaccessible for $\mathcal{E}$.

In case the system is classical, an ideal first kind measurement which gives a positive response does not change the system. This has been mentioned above and now points to the fact that in such a “static” case, the Sasaki hook reduces itself into a material implication. This however is not surprising. Indeed, it has been stated in [12] that all five “candidates” for an implication within quantum logic which satisfy the strengthened law of entailment, under which we find the Sasaki hook, are equal to a material implication if and only if the lattice is boolean. The Sasaki hook is in this respect even more special than the others since it approaches the material implication of classical logic more closely. Indeed, it also satisfies a weaker condition namely, $a \wedge x \leq b \Leftrightarrow x \leq (a \xrightarrow{S} b)$ if $a$ and $b$ are compatible. This condition is called the weak import-export condition in [12] and has been written as: $(a \xrightarrow{S} b) = a' \vee b$ if $a \leftrightarrow b$ in [15] where it is called the *locally boolean*-condition. Concerning our analysis of the Sasaki hook within OQL, this is as far as we can go. Within the framework of Dynamic OQL (see for instance

[9, 31]) and more exactly in [11], it is shown that the Sasaki hook has a fundamental causal-assigning nature and gives rise to a dynamic implication for which a deduction theorem holds with respect to a dynamic conjunction. In the following section we concentrate on the causal assignment interpretation of the Sasaki hook and the implicit connection with its counterfactual nature.

## 5. *The Causal Assignment Interpretation*

For a given $a \in \mathcal{L}$, we conceive of $(a \xrightarrow{S} -)$ as assigning to some property $b \in \mathcal{L}$ the weakest cause $(a \xrightarrow{S} b)$ before the measurement of actuality of $b$ after the measurement. This is to say that the whole of $(a \xrightarrow{S} b)$ is the weakest property which, if it is actual in some state $p$, *guarantees* that $b$ will be actual under the condition that we obtain a positive response from performing the ideal first kind measurement $\alpha$ with $\zeta(\alpha) = a$. Indeed, as we argue below, we have good reasons to accept that the property $(a \xrightarrow{S} b)$ stands for "guaranteeing the actuality of $b$ under some condition which positively tests for $a$". As such we know that when the property $(a \xrightarrow{S} b)$ is actual, also the actuality of $b$ is guaranteed in case we fulfill the necessary condition. When the property $(a \xrightarrow{S} b)$ is potential, the actuality of $b$ is not guaranteed.

Our argument is in first instance operational and focuses on a certain kind of question which combines so-called inductions with questions. More explicitly, the notion of an induction $e$ has been introduced in [3]:

> "We talk about an induction in case of an externally imposed change of a particular entity $\Xi$. This change might modify the collection of actual properties of $\Xi$, i.e., its state, as well as the whole collection of properties, i.e., $\Xi$ itself; in the case that $\Xi$ is preserved with certainty we speak of a soft induction, otherwise of a hard induction." (p. 556)

A specific combination of soft inductions and questions, following a similar idea in [13], has been defined in [10]:

> $e.\beta$ is the question consisting of "first executing the induction $e$ and then performing the question $\beta$", the outcome of the question $e.\beta$ being the one thus obtained for $\beta$.

Given the meaning of a true question, we call $e.\beta$ true if it is sure that $\beta$ would be true should we perform induction $e$. Since to a true question there corresponds an actual property, we introduce the notation $e.b$ for the property which stands for "guaranteeing the actuality of $b$". For more details on the introduction of soft inductions as actions on properties we refer for instance to [31]. Applying these ideas to the context of Sasaki hooks and projections we have to work with an ideal first kind measurement $\varphi_a$ which induces property $a$ in case of a positive response.

> $\varphi_a.\beta$ is the question consisting of the experimental procedure: "first execute $\varphi_a$ so that a positive response is obtained, then perform the question $\beta$" and the rule: "the outcome of the question $\varphi_a.\beta$ is the one obtained for $\beta$".

More specifically, from "$\varphi_a.\beta$ is true for a system in a certain realization" immediately follows that "$\beta$ is true for that system after positively performing $\varphi_a$". Exactly this expression forms the underlying idea of introducing a formal causal relation $\overset{\varphi_a}{\rightsquigarrow}$ with respect to the action $\varphi_a$ on questions. Here $\varphi_a.\beta \overset{\varphi_a}{\rightsquigarrow} \beta$ expresses from the point of view before we perform $\varphi_a$ that "if $\varphi_a.\beta$ is true then $\beta$ will be true, after positively performing $\varphi_a$". Under acceptance of Hardegree's interpretation and the link we proposed with OQL, we may confirm that the Sasaki hook $a \overset{S}{\rightarrow} b$ can be operationally approached by means of the property $\zeta(\varphi_a.\beta) = \varphi_a.b$. We explain this by means of the mappings expressing the Sasaki duality as conceivable within Dynamic OQL, for the purpose of which we lift the introduced causal relation to the level of properties: $\overset{\varphi_a}{\rightsquigarrow} \subseteq \mathcal{L}_1 \times \mathcal{L}_2$. This causal relation allows us to express in the specific case of $c \overset{\varphi_a}{\rightsquigarrow} b$ that the actuality of $c$ guarantees that after positively performing $\varphi_a$, $b$ will be actual. Remark that for the sake of clarity we did not turn our attention to the earlier mentioned problem of inaccessible states, which also occurs here. In particular we should for instance allow a formal extension of the causal relation so that $a' \overset{\varphi_a}{\rightsquigarrow} 0$ captures the impossible transition, i.e. when property $a'$ is actual in the state in which one wants to induce property $a$. It is plausible to interpret the impossible transition as the case in which performing $\varphi_a$ destroys the system. Note that this interpretation allows for a formal characterization by means of a top-element extension instead of working with an absurd element 0, for details we refer to [10, 32]. We further allow ourselves to abstract over the case of impossible transitions when introducing the following maps from an operational perspective:

1) Sasaki projection:
$\varphi_a^* - : \mathcal{L}_1 \to \mathcal{L}_2 : c \mapsto \bigwedge\{b \in \mathcal{L}_2 \mid c \overset{\varphi_a}{\rightsquigarrow} b\}$
2) Sasaki hook:
$a \overset{S}{\rightarrow} - : \mathcal{L}_2 \to \mathcal{L}_1 : b \mapsto \varphi_a.b = \bigvee\{c \in \mathcal{L}_1 \mid c \overset{\varphi_a}{\rightsquigarrow} b\}$

Following [10, 11], we see that $(a \stackrel{S}{\rightarrow} -)$ assigns to any property $b \in \mathcal{L}_2$ the weakest property $(a \stackrel{S}{\rightarrow} b) \in \mathcal{L}_1$ whose actuality before positively performing $\varphi_a$ guarantees actuality of $b$ afterwards. Similarly, $\varphi_a^*-$ assigns to any property $c \in \mathcal{L}_1$ the strongest property $\varphi_a^*(c) \in \mathcal{L}_2$ of which actuality after positively performing $\varphi_a$ is due to the actuality of $c$ beforehand. Since $\varphi_a^*(a \stackrel{S}{\rightarrow} b) \leq b$ and $c \leq (a \stackrel{S}{\rightarrow} \varphi_a^*(c))$ we have $\varphi_a^*- \dashv (a \stackrel{S}{\rightarrow} -)$, which expresses the causal duality for the case of the action called $\varphi_a$.[8] Causal duality has here been applied to specific temporal processes and underlies the motivation for the causal assignment interpretation of the Sasaki hook. Note that the causal duality proposal in [10] took place in a more general setting and applies also to situations of compoundness [6].

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