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MODELLING TRUTHMAKING

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According to one tradition in realist philosophy, 'truthmaking' amounts to necessitation. That is, an object x is a truthmaker for the claim A if x exists, and the existence of x necessitates the truth of A. In symbols: $E!x \wedge (E!x \Rightarrow A)$.

I argued in my paper "Truthmakers, Entailment and Necessity" [14], that if we wish to use this account of truthmaking, we ought understand the entailment connective "$\Rightarrow$" in such a claim as a *relevant* entailment, in the tradition of Anderson and Belnap and their co-workers [1, 2, 8, 11]. Furthermore, I proposed a number of theses about truthmaking as necessitation. The most controversial of these is the disjunction thesis: x makes a disjunction $A \vee B$ true if and only if it makes one of the disjuncts (A or B) true.

That paper left one important task unfinished. I did not explain how the theses about truthmaking could be true together. In this paper I give a consistency proof, by providing a *model* for the theses of truthmaking in my earlier paper. This result does two things. First, it shows that the theses of truthmaking are jointly consistent. Second, it provides an independently philosophically motivated formal model for relevant logics in the 'possible worlds' tradition of Routley and Meyer [8, 16, 17].

1. *The Theses*

In an earlier paper, "Truthmakers, Entailment and Necessity" [14], I introduced and defended a number of theses about truthmaking and its connection with an account of relevant entailment. It is one thing to introduce a number of theses and to defend them. It is another to show that these theses are *consistent*. In this section I will introduce and explain the motivation for these theses.

The first thesis introduces the account of truthmaking as *necessitation*.

- x makes A true if and only if x exists, and the existence of x entails the truth of A. In symbols, x makes A true if and only if $E!x \wedge (E!x \Rightarrow A)$.

Truthmaking is a relation between objects on the one hand and claims on the other. An object makes a claim true just when the existence of the object *entails* the truth of the claim. In "Truthmakers, Entailment and Necessity" I argued that this entailment should be interpreted as *relevant* entailment. This means that for $A \Rightarrow B$ to hold, there must (at the very least) be some kind of *connection* between A and B. An entailment is not given *simply* because the consequent B is necessary, or because the antecedent A is impossible. We will look more at the notion of relevant entailment in a later section. For the moment, it will suffice to use an intuitive notion of entailment to understand this account.

The next thesis holds that anything true is made true by something.

- A is true if and only if there is something which makes A true.

One half of this biconditional is trivial. If x makes A true, then since x exists and the existence of x entails A, we must have A true. The other half is much more controversial. This is the thesis which connects truth tightly to *ontology*. If something is true, there is some *thing* which makes it true. Many have thought that this is far too strong. What, after all, makes true negative claims, universal claims, necessities or possibilities? There is much to be done to argue for this strong truthmaking claim, but I will not do it here. I intend to merely discuss its *consistency*.

Our next thesis connects truthmaking and conjunction.

- x makes $A \wedge B$ true if and only if x makes A true and x makes B true.

This thesis is an immediate consequence of the previous thesis and behaviour of entailment. If $E!x \Rightarrow A \wedge B$, then $E!x \Rightarrow A$ and $E!x \Rightarrow B$ and conversely. This conjunction thesis is not only an immediate consequence of the truthmaking definition, but it is also independently plausible on many accounts of truthmaking. If x makes a conjunction true, it makes both conjuncts true. But to make a conjunction true it suffices to make both conjuncts true. What else could there be to do?

More controversial by far is the thesis connecting truthmaking and *disjunction*.

- x makes $A \vee B$ true if and only if x makes A true or x makes B true.

This thesis is by no means *obvious*. It does not naturally fall out of the conception of truthmaking as necessitation. To be more precise, one half of it does fall out of the necessitation conception of truthmaking (right-to-left) for if $E!x \Rightarrow A$ then $E!x \Rightarrow A \vee B$, and similarly, if $E!x \Rightarrow B$ then $E!x \Rightarrow A \vee B$. So, if x makes A true, or if x makes B true, then x must make $A \vee B$ true too. However, the left-to-right direction is much more problematic. There may well be some way to make $A \vee B$ true which does not (in and of itself) make A true or make B true.

However, the disjunction thesis does have a certain plausibility. In what way could something make $A \vee B$ true without making A true or making B true? In particular, if some truthmaker ensures the truth of a disjunction $A \vee B$, then there must be *some* thing which either makes A true or makes B true. After all, if $A \vee B$ is true then either A is true, or B is true.[1] Again, however, we will not spend more time on this thesis to defend it. That was done in my previous paper. I will simply examine its consistency with the other theses.[2]

The next thesis involves negation.

- Something makes $\sim A$ true if and only if nothing makes A true.

Note that this thesis does *not* say that a truthmaker makes $\sim A$ true if and only if that truthmaker does not make A true. That would be a swift ticket to *truthmaker monism*, the doctrine that all truthmakers make true every truth. No, according to our theses, truthmakers can allow for a degree of division of labour. If x does not make A true, another truthmaker y might well do the job. Our negation thesis demands that in this case, x (and any other truthmaker) had better not make $\sim A$ true. And conversely, if no truthmaker makes A true, then *some* truthmaker had better step up to the plate to make $\sim A$ true.

This thesis actually *follows* from the previous theses and the claims that $A \vee \sim A$ is always true and that $A \wedge \sim A$ is never true. I include it here simply for completeness' sake.

The final thesis was added to "Truthmakers, Entailment and Necessity" as an afterthought. It connects truthmaking back to entailment.

- A (relevantly) entails B if and only if necessarily, for each x, if x makes A true then x makes B true.

This thesis was designed to give an account of relevant entailment to those who find the notion difficult to understand. It closes the circle by defining

[1] A similar tradition is the *situation theory* of Barwise and Perry [4]. In this tradition, situations are restricted pieces of the world which determine all that is *inside* them. If in this situation the milk is in the refrigerator or on the table, then in this situation the milk is in the refrigerator, or in this situation it is on the table.

[2] I will not spend much time considering quantifiers. However, the obvious generalisation of the disjunction thesis to the existential quantifier is worth considering. Is it true that if x makes true $\exists v F(v)$ then there is some object a such that x makes $F(a)$ true? Our model will validate this condition too. However, I am not at all sure whether this ought be defended.

relevant entailment in terms of truthmaking. This thesis was added as an afterthought, and it is by no means clear that it is consistent with what has gone before it. I have defined truthmaking in terms of entailment. Is is consistent with this definition to define entailment in terms of truthmaking? I will show that this is in fact the case. Our theses are jointly consistent, for they have a model. It is to this model which we will turn.

2. *Flat Worlds, and Regions*

Consider a Euclidean plane infinite in all directions, marked off into squares in a regular grid. A square can be inhabited or uninhabited. Here is how we will represent an inhabited square and an uninhabited square respectively.

$$\blacksquare \qquad \square$$

A *world* is any such plane, in which each square is either inhabited or uninhabited. So, here is what a *part* of a world might look like:

$$\begin{array}{c}\square\square\square\square\square\\ \square\blacksquare\blacksquare\blacksquare\square\\ \square\blacksquare\square\blacksquare\square\\ \square\blacksquare\blacksquare\blacksquare\square\\ \square\square\square\square\square\end{array}$$

In this part of a world, eight squares are *inhabited* and seventeen are *uninhabited*. This part of a world covers twenty-five squares. We will call parts of worlds *regions*. Regions will function as *truthmakers* in our model.

A *world* can be represented as a function $w : \mathbb{Z} \times \mathbb{Z} \to \{\blacksquare, \square\}$. This function maps every coordinate in the integer plane to a value $\blacksquare$ or $\square$. If you prefer sets to functions, a world is a set of coordinateŰvalue pairs

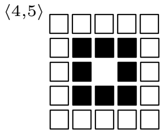
$$\{\langle\langle 0,0\rangle, \blacksquare\rangle, \langle\langle 0,1\rangle, \square\rangle, \langle\langle -1,0\rangle, \blacksquare\rangle, ...\}$$

in which every coordinate features once and once only. I will primarily use functions to model worlds in what follows.

A region then can be modelled by a *partial function* $r : \mathbb{Z} \times \mathbb{Z} \to \{\blacksquare, \square\}$. That is, it assigns on/off values to *some* integer coordinates.[3] Again, regions can be represented by sets, rather than partial functions if you prefer. Nothing hangs on the means of representation. In fact, from now, I will simply *identify* worlds and regions with the functions which represent them.

We can *picture* regions by picking out the coordinates of one point of the region, as follows.

[3] In fact, we will allow the *empty* region, which assigns no values at all, to be a region. This is purely a matter of convenience, when it comes to modelling entailment. Nothing of significance hangs on this matter.

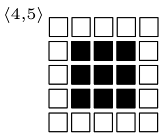


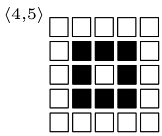
The coordinate indicates the location of the part in the larger 'world'. In this region, $r(4,5) = \square$, and $r(5,4) = \blacksquare$, but $r(6,3)$ is undefined. This region has a *hole* in it. That is, $r(6,3)$ is undefined. It is not empty, and it is not full. The point $\langle 6,3\rangle$ does not feature in the region r. For our purposes regions can contain holes, and they may even be disconnected. There is no requirement in this model that regions be connected or *natural* in any way.

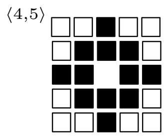
A region r *is a part of* another region s if and only if $r(n,m) = s(n,m)$ wherever $r(n,m)$ is defined. We write this as

$$r \sqsubseteq s$$

If regions were represented as sets, this inclusion relation would simply be subsethood. The region depicted above is a part of the first two regions below, but not the third.







The first two regions expand on r by including its central hole. In the first, the hole is included and filled, in the second, it is included and empty. In the third region, four points are filled which were empty in r (namely, $\langle 6,5\rangle$, $\langle 4,3\rangle$, $\langle 8,3\rangle$, and $\langle 6,1\rangle$).

3. *Introducing Our Language*

We will now use this *ontology* of regions and worlds to provide the underpinnings for a *language*, in which we will be able to express our claims about truthmaking. Our language will be a simple first-order one with conjunction, disjunction, negation and quantifiers, together with the existence predicate $E!$, a relevant entailment ($\Rightarrow$) and necessity ($\Box$). This will enable us to formalise each of our theses.

We will include a name $\underline{r}$ for *each* region r. Regions will count as truthmakers in this ontology, and we will need to be able to refer to them in order to state our theses about truthmaking. Our language is constructed from these basic items in the usual way.[4]

Our model will be based on a fundamental relationship between regions and sentences. This relationship will be expressed in the *metalanguage*. We will write

$$r \Vdash A$$

to indicate that according to the region r, A holds. This is not a statement *in* our language. It is a statement *about* the relationship between our ontology and our language. However, it has an *analogue* in our language

$$E!\underline{r} \Rightarrow A$$

This is a sentence of our language, and we will show that this sentence will be true *in* our model (in a sense to be defined soon) if and only if $r \Vdash A$ is true *of* our model.

The relationship $\Vdash$ is one of *weak* truthmaking. If $r \Vdash A$, then *according to* r, A *holds*. That does not mean, in and of itself (in our model) that A holds. For A to hold we need r to *exist*, or be *actual*. The *existence* of the truthmaker is needed for *strong* truthmaking, then notion of primary interest to us.

A fundamental *constraint* on weak truthmaking is the HEREDITARY CONDITION.

- If $r \sqsubseteq s$ then $r \Vdash A$ only if $s \Vdash A$.

According to this condition, if r is contained in s, then anything made true by r is also made true by s. Truth *expands* as regions expand. This is an important restriction on the kinds of propositions we can consider, or more accurately, it is a restriction on how we model them. For example, the proposition 'There is are no inhabited points here' should not be modelled as a simple negative existential, which is true in a small (uninhabited) region and false in a larger (inhabited) one. That would fall foul of the hereditary condition. No, to model this, the proposition must make reference to the region involved. The claim that there is no beer in r might be true in r, and still be true in a larger, beer-including region s, which contains r.[5]

We will go on to define the behaviour of the most important predicate in our language, the existence or actuality predicate.

[4] I prefer the definition according to which there are no formulae with free variables. We have names for each region, so we will be able to give *substitutional* clauses for the quantifiers.

[5] For more on the distinction between *persistent* and *non-persistent* propositions in the context of situation theory, see Barwise's "Branchpoints in Situation Theory" [3].

- $r \Vdash E!\underline{s}$ if and only if $s \sqsubseteq r$.

This definition of the extension of the existence predicate is a natural one. According to r, s exists (or is actual) just when s is a part of the region r. The region r "knows about" all and only those regions inside it. This definition satisfies the hereditary requirement. If s exists according to r, then it exists according to any larger region.

If you are interested in adding other predicates to our language, the only constraint you need consider is the hereditary requirement. For example, for each coordinate $\langle m, n\rangle$ you can add the simple proposition (construed as a zero-place predicate) $\mathsf{Inhabited}(m, n)$ such that

- $r \Vdash \mathsf{Inhabited}(m, n)$ if and only if $r(m, n) = \blacksquare$.

This seems to be the appropriate interpretation of inhabitation.

We end this section with the other simple recursive clauses for truth-in-regions, those for conjunction and disjunction. These have the obvious definitions.

- $r \Vdash A \wedge B$ if and only if $r \Vdash A$ and $r \Vdash B$.
- $r \Vdash A \vee B$ if and only if $r \Vdash A$ or $r \Vdash B$.

These connectives do not fall foul of the hereditary requirement. If A and B satisfy heredity, then so will their conjunction and disjunction.

4. *Compatibility and Negation*

It is a little more difficult to define the semantics of negation in a way which also satisfies the hereditary requirement. It is no use to require that $r \Vdash {\sim}A$ if and only if $r \nVdash A$. That will not do, unless all regions which are part of the one world agree on all propositions, and that leaves us with an uninteresting notion of truthmaking. One must look to alternative accounts of negation.

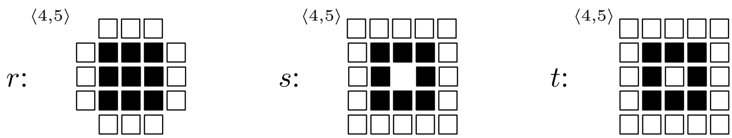
One such alternative account is to define the *negative extension* of a predicate (where it is *false*) along with its positive extension (where it is true). Then one defines truth and falsity of more complex propositions together. This is the approach of Nelson's constructible falsity [12], Dunn's semantics for first-degree entailment [7], and Barwise and Perry's situation theory [4]. As such it has a long heritage. However, we will not take this approach here. Instead of defining positive and negative extensions and having *two* recursive clauses for each connective or operator (which is easy for conjunction and disjunction, but more difficult for the quantifiers, modality and entailment) we will define negation by way of the relation of *compatibility* between regions [9, 10, 15].

What is it for two regions to be compatible? For us, it will be that the two regions r and s share no points of *disagreement*. The regions r and s are compatible (which we write as 'rCs') just when there is no pair $\langle m, n\rangle$ of coordinates such that according to one region, $\langle m, n\rangle$ is inhabited by ■, and according to the other, that point is inhabited by □. That is, there is no point at which r and s *disagree*. Given this definition of compatibility, the account of negation is straightforward.

- $r \Vdash {\sim}A$ if and only if for each s where rCs, $s \nVdash A$.

That is, according to r, A is *not* true just when for every *compatible* s, A does not hold in s. Or equivalently, ${\sim}A$ is true in r just when any region in which A holds is incompatible with r.

Here are three regions, r, s and t.



The region r is compatible with region s, but not with t (r and t disagree on the central point in the square). Region s, on the other hand, is compatible with both r and t.

This clause for negation allows regions to be *incomplete*. If $s \sqsubseteq r$, but r is strictly larger than s (as in the case above) then $s \nVdash E!\underline{r}$, but on the other hand, $s \nVdash {\sim}E!\underline{r}$, for there is a region compatible with s (namely, r) in which $E!\underline{r}$ holds. Therefore $s \nVdash E!\underline{r} \vee {\sim}E!\underline{r}$. Similarly, $r \nVdash \mathsf{Inhabited}(4,5) \vee {\sim}\mathsf{Inhabited}(4,5)$.[6]

However, the negation operator in these models is not really a non-classical one. Negation operates classically at *worlds*, in which we have a totality of information, and no "gaps". For any world w, the only regions r such that wCr are those r which are a *part* of w. It follows that worlds are consistent and complete, as you would expect. We can reason like this: if $w \Vdash {\sim}A$, then since wCw, $w \nVdash A$. Conversely, if $w \nVdash A$, then for every $r \sqsubseteq w$, $r \nVdash A$ (by heredity). It follows that for every r where wCr, $r \nVdash A$, which gives us $w \Vdash {\sim}A$ as desired. So, worlds satisfy the *classicality* condition

$$w \Vdash {\sim}A \text{ if and only if } w \nVdash A$$

Worlds are completely classical, and negation works as you would expect there. It is not surprising, however, that negation behaves a little oddly *elsewhere*. For example, in our model, although r need not make each $A \vee {\sim}A$

[6] Note that according to this definition, $r \Vdash {\sim}\mathsf{Inhabited}(n, m)$ if and only if $r(n, m) =$ □, as one would hope.

true, it does make $\sim\sim(A \vee \sim A)$ true. For there is *no* region s at which $\sim(A \vee \sim A)$ holds (all regions are consistent: for any r, rCr). Therefore for all regions r, $r \Vdash \sim\sim(A \vee \sim A)$. It follows that double negation elimination fails to be a valid consequence on what is true in a *region*.[7]

I will not discuss the desirability or otherwise of this result at this point. I will merely end this section *indicating* how it can be surmounted and how the validity of the rule of double negation elimination (and all the de Morgan laws) can be restored. The way ahead is to allow *inconsistent* as well as incomplete regions. A region can now be represented as a *relation* between coordinates and values instead of a function.[8] Now a region can underdetermine, uniquely determine, or *overdetermine* the inhabitant of any point. Instead, then of writing '$r(m, n) = \blacksquare$' to indicate that the value of r at $\langle m, n\rangle$ is $\blacksquare$, we will write '$\blacksquare r(m, n)$' to indicate that $\blacksquare$ is *one* value of r at $\langle m, n\rangle$. A region r is *consistent* if and only if whenever $\blacksquare r(m, n)$, it is not the case that $\square r(m, n)$. A region r is *complete* if and only if whenever it is not the case that $\square r(m, n)$ it is the case that $\blacksquare r(m, n)$. Worlds are the complete, consistent regions.

Inclusion of relations is defined as before: $r \sqsubseteq s$ if wherever r says there is a $\blacksquare$ so does s, and wherever r says there is a $\square$ so does s. Note that any region included in a consistent region is itself consistent, and that any region including an inconsistent region is itself, inconsistent. Worlds now are *not* at the top of the inclusion chain. They are themselves contained in inconsistent regions which extend them by being *confused* about points.

Compatibility, then, is defined just as before. The region r is compatible with the region s just when there is no point (m, n) where one region takes there to be (at least) a $\blacksquare$ and the other takes there to be (at least) a $\square$. Now a region may well not be compatible with *itself.* (The only way for a region to disagree with itself is for it to be inconsistent, of course.) Perhaps more surprisingly, an inconsistent region can be compatible with other regions. If r is confused about $\langle m, n\rangle$, it may well be compatible with s, if s says nothing about that point. If that is the case, r and s do not disagree. They are compatible.[9]

[7] However, it is of course valid in each *world*. If $w \Vdash \sim\sim A$ then $w \nVdash \sim A$, and by completeness, $w \Vdash A$.

[8] Or, if you prefer, a completely *arbitrary* set of coordinate–value pairs.

[9] This means that the compatibility of r and s does *not* mean that they have a consistent extension. I might well not disagree with you, but it does not follow that our beliefs can be fused into a consistent whole, for either (or both) of us might be inconsistent about something, about which the other has no opinions.

From all of this, it follows that negation satisfies double negation elimination (and introduction) and the de Morgan laws. The result is not a complex one. The idea is that now, each point r has a *maximal compatible partner* r^*. The region r^* is ■ where r is ■ only, □ where r is □ only, it is inconsistent where r is incomplete, and incomplete where r is inconsistent. It follows that rCr^*, and furthermore, if rCs, then $s \sqsubseteq r^*$. The region r^* "wraps up" all of the regions compatible with r. By definition, it is not hard to see that $r^{**} = r$, and that $r \Vdash {\sim}A$ if and only if $r^* \nVdash A$. It follows that if $r \Vdash {\sim}{\sim}A$, then $r^* \nVdash {\sim}A$, and then that $r = r^{**} \Vdash A$. Double negation elimination is preserved.

There is more that can be said in the defence of the use of inconsistent regions. I have done this elsewhere [15], and I will not labour the point here. Suffice to say, again, that negation treated in this way is not particularly nonclassical. Worlds are the regions w such that $w = w^*$, and negation works completely classically in each world. Furthermore, the inconsistent regions do not exist (are not actual) in any world. It will follow (in the next section, when we define necessity) that they are necessarily nonexistent. They are *impossibilia*.

As attractive as inconsistent regions are for our logic, and as understandable as they are in these models, you do not need to accept them to do truthmaking. In the next sections, we not presuppose the acceptance of inconsistent regions, or, for that matter, will we do anything which will rule them out.

Before moving on to the rest of our language, we should pause to note that negation, defined in terms of compatibility, satisfies our hereditary condition. If r and s are compatible, then any region inside r is compatible with s. Therefore, if $r \nVdash {\sim}A$, there is some compatible s where $s \Vdash A$. It follows that if $r' \sqsubseteq r$, $r' \nVdash {\sim}A$ too. We have just proved the *contraposition* of the hereditary condition. If $r' \Vdash {\sim}A$ and $r' \sqsubseteq r$, it follows that $r \Vdash {\sim}A$ too.

5. *Quantifiers*

We must model the quantifiers, for they appear in our last thesis about truthmaking. In our language we have a name for *every* region, so quantification can be done *substitutionally*. We have the following clauses:

- $r \Vdash (\forall x)A(x)$ if and only if $r \Vdash A(\underline{s})$ for each region s.
- $r \Vdash (\exists x)A(x)$ if and only if $r \Vdash A(\underline{s})$ for some region s.

These quantifiers are not particularly odd, except for the fact that they allow for quantification over *nonexistent* objects. For example, take the regions r, s and t from page 218. Regions r and t are incompatible, so it follows that any region compatible with r, cannot contain t. Therefore $r \Vdash {\sim}E!\underline{t}$, and by the clause for the particular (*existential*?) quantifier, $r \Vdash (\exists x){\sim}E!x$. According to r, some thing does not exist.

Is this substitutionally defined quantifier inappropriate for our last truthmaking thesis? This thesis states, if you recall, that A relevantly entails B just when, of necessity, any truthmaker for A is a truthmaker for B. Recall that *strong* truthmaking is at issue here, for which *existence* is necessary. If r is a truthmaker for A in *this* sense, we require that r exist. So, this clause will have to be formalised using $E!$ to restrict the scope of the quantifier in question. No "funny business" with nonexistent regions will interfere in the interpretation of this clause, and the substitutionally defined quantifier seems appropriate.

6. *Necessity and Entailment*

From this point on, the evaluation conditions are much more *tentative*. We have a very good idea of what it is for a claim of the form $\Box A$ to be *true*. It is for it to be true in any different possibility (or *world*). We have much less idea of what *makes* such a claim true. Or in our models, we have much less idea of what is for $\Box A$ to be true in a *region*. Our certainty fades even earlier when it comes to the truth and the truthmakers of *relevant entailment*. As a result, the evaluation conditions in this section should be taken as providing a *model* which verifies the truthmaking theses, but not as in any sense an *interpretation* which gives an account of the meanings of modal claims.

Consider necessity first. We wish to give an account of when $r \Vdash \Box A$. There seem to be at least two reasonable constraints to any such account. First, it seems quite plausible to hold that $w \Vdash \Box A$ if and only if in each world w', $w' \Vdash A$. This is the traditional modal account of necessity. According to *this* world, A is necessary, just when A is true in *each* world. This is the simple possible worlds account of necessity, and it yields the logic S5.

This constraint gives us more than you might think. In conjunction with the hereditary condition, it shows that in any region r which is included in a world, $\Box A$ is true *only if* A is true in each world. Regions cannot hold that anything is necessary which is not true in each world.

That is one constraint. The second reasonable constraint is that if $r \Vdash \Box A$, then $r \Vdash A$. That is, if r takes A to be *necessary*, then r itself also takes A to be true. This seems like a reasonable constraint on an interpretation of necessity. It follows from these two constraints (with all that has gone before them) that we should *not* require that $r \Vdash \Box A$ if and only if A is true in each *world*. For $A \vee {\sim}A$ is true in each world, and it would follow that $r \Vdash A \vee {\sim}A$ for *any* region r, giving either $r \Vdash A$ or $r \Vdash {\sim}A$, and truthmaker monism.

So, the truth of $\Box A$ (in a region) does not amount simply to its truth in all worlds. That will suffice for its truth in all *worlds*, but for regions smaller than worlds, we require *more* to show that a proposition is necessary. What can we do to give an account of this?

One approach is to keep the idea of an *accessibility* relation, but to expand it to take regions into account as well as worlds. We will have $r \Vdash \Box A$ if and only if for each s *accessible from* r, $s \Vdash A$. What will such an accessibility relation look like? From our first constraint, we require that all worlds (and only worlds) are accessible from worlds. From our second constraint, we require that regions be accessible from themselves. There are a multitude of candidate accessibility satisfying these two constraints. We will consider just one:

> The region s is *accessible from* r just when r and s *share areas*. That is, when r and s are *defined* on exactly the same points.

We write this as '$r \approx s$'. If we have just consistent regions in our model, then worlds and only worlds are accessible from worlds.[10] For smaller regions r, worlds are not accessible, but any other region on the same area is. For example, the first and the second regions on page 215 are mutually accessible. The third region is not accessible from either of the others, because it is on a strictly smaller area. So, we have the following modelling condition on necessities.

- $r \Vdash \Box A$ if and only if for each s where $r \approx s$, $s \Vdash A$.

The resulting picture is this: regions "know about" their areas. Worlds, have *total* areas, so they know about everything. Any modal necessity at all is true in each and every world (this is S5). However, not all modal truths are true in each *region*. However, some are. For example, $s \Vdash E!\underline{s}$ (in s, s exists). On any *other* region r on the same area, s does not exist. That is, $r \Vdash {\sim}E!\underline{s}$. Any *other* region on the same area is incompatible with s. Therefore, for all regions r on the same area as s, we have $r \Vdash E!\underline{s} \vee {\sim}E!\underline{s}$, and as a result, $s \Vdash \Box(E!\underline{s} \vee {\sim}E!\underline{s})$. This instance of the law of the excluded middle is necessary at a region more restricted than an entire world.

We had, in fact, one more constraint for our interpretation of necessity: the hereditary constraint. We must verify that if $r \sqsubseteq s$, and $r \Vdash \Box A$, then

[10] If we have inconsistent regions, then any *complete* region (including inconsistent ones) will be accessible from a world, but this is no problem for the evaluation of modal claims. For each inconsistent region has consistent regions contained in it, and as a result we will still have $w \Vdash \Box A$ only if $w' \Vdash A$ for each world w'.

$s \Vdash \Box A$. But this is not difficult. If $r \Vdash \Box A$, then for any r' where $r \approx r'$, we have $r' \Vdash A$. Now if we have some s' where $s \approx s'$, we can find some r' where $r' \sqsubseteq s'$ and $r' \approx r$. (How? Consider the intersection of s' with the area covered by r. The area covered by r is included in that covered by s, so such an intersection exists.) Since $r \Vdash \Box A$, we have $r' \Vdash A$, and hence $s' \Vdash A$. This shows that $s \Vdash \Box A$, as desired. The hereditary condition is satisfied.

We have, therefore, a notion of necessity in our models. It acts in a completely orthodox way when restricted to worlds. We will use this account of necessity in verifying our truthmaking theses.[11]

Necessity is a simple matter, when compared with relevant entailment. Studies of the semantics of entailment have resulted in ideas of great formal elegance, but the structures studied have not found universal appeal [5, 6, 13]. As a result, the account given here is doubly tentative.

We will define entailment in a similar manner to necessity. The idea of the *area* of a region will play an important role. Again, we have constraints on appropriate interpretations of entailment. First, we want worlds to make $A \Rightarrow B$ true just when any region in which A is true is one in which B is true. This will make $\Rightarrow$ a *relevant* entailment. Second, we would like to allow regions to somehow be more *discriminating* than worlds in making entailments true. Regions may be appropriately *local*. Third, we must satisfy the hereditary constraint. There is at least *one* way to model entailment which meets these criteria. The crux of the definition is found in the notion of a *restriction*. Given regions s and r, the *restriction* of s to the *area* of r, is the region $s|_r$ is the subregion of s which covers only the area covered by r. If there is no such subregion (if r and s do not overlap in area) then $s|_r$ is the empty region.[12] Here are some facts about *restriction*.

- If $r \sqsubseteq s$ then $r|_s = r$ and $s|_r = r$. If w is a world, $r|_w = r$.

[11] What of possibility? There are two ways to go. One is to define $\Diamond$ as $\sim\Box\sim$, and the other is to use the existential quantifier and the same accessibility relation in the standard modal definition. In the case of worlds, these definitions agree. In the case of regions they differ. If we have inconsistent regions, $r \Vdash \sim\Box\sim A$ if and only if there is some s where $r^* \approx s^*$ where $s \Vdash A$. In our models, we might have $r \approx s$ without $r^* \approx s^*$ (if r is a consistent subregion of s, for example) or *vice versa*. It is unclear what should be said about this difference. One radical but plausible proposal is to say that $\Diamond$ and $\sim\Box\sim$ are simply *not equivalent*. Of course, they agree at all *worlds* but not necessarily in all *regions*.

[12] It is possible to do away with the empty region in what follows, at the cost of complicating our account of entailment a little. Instead of talking of $s|_r$ we talk of the *three-place relation* $s|_r \sqsubseteq t$ (abusing notation somewhat), which holds if and only if each point $\langle m, n\rangle$ which is assigned a value by s *and* in the area of r is assigned the same value by t. Then $r \Vdash A \Rightarrow B$ if and only if for each s, t where $s|_r \sqsubseteq t$, if $s \Vdash A$ then $t \Vdash B$.

Restriction works like intersection when the two regions are ordered. Furthermore, the restriction of any region to a *world* (any world) is that region itself.

- If $r \sqsubseteq r'$ then $s|_r \sqsubseteq s|'_r$ and $r|_s \sqsubseteq r'|_s$.

Restriction is *monotonic* in both positions. This will be important when verifying that entailment satisfies the hereditary constraint.

Given this account of restriction, we can define entailment as follows:

- $r \Vdash A \Rightarrow B$ if and only if for each s where $s \Vdash A$, $s|_r \Vdash B$.

That is, $A \Rightarrow B$ is true at r just when *wherever* A is true, B is true at the restriction of that region to r. This meets our constraints. First of all, $w \Vdash A \Rightarrow B$ if and only if for each s where $s \Vdash A$, we have $s = s|_w \Vdash B$. An entailment is true in a *world* just when in any region in which the antecedent is true, so is the consequent.

Second, regions can indeed be more local with respect to entailments. For example, $w \Vdash E!\underline{r} \Rightarrow E!\underline{r}$, but this true entailment is not verified in *every* region. We have $s \Vdash E!\underline{r} \Rightarrow E!\underline{r}$ only if whenever $t \Vdash E!\underline{r}$, $t|_s \Vdash E!\underline{r}$. Now, $t \Vdash E!\underline{r}$ if and only if $r \sqsubseteq t$. Therefore $s \Vdash E!\underline{r} \Rightarrow E!\underline{r}$ if and only if $r|_s \Vdash E!\underline{r}$. This will obtain only if s covers at least the region of r. This need not be the case. So, entailments may well be local matters.

Third, entailment does indeed satisfy the hereditary constraint. If $r \Vdash A \Rightarrow B$, then if $r \sqsubseteq r'$, and $s \Vdash A$, then since $s|_r \Vdash B$ we also have $s|'_r \Vdash B$ (by the monotonicity of restriction). Entailments are preserved as you go up the inclusion ordering.

Now we can look at the *internal* reading of truthmaking. The internal reading of $s \Vdash A$ is the sentence $E!\underline{s} \Rightarrow A$. When does $r \Vdash E!\underline{s} \Rightarrow A$? We can reason as follows: $r \Vdash E!\underline{s} \Rightarrow A$ if and only if for each t where $t \Vdash E!\underline{s}$, $t|_r \Vdash A$. But $t \Vdash E!\underline{s}$ if and only if $s \sqsubseteq t$. So, by monotonicity, $t|_r \Vdash A$ whenever $t \Vdash E!\underline{s}$ if and only if $s|_r \Vdash A$. So, *according to* r, s makes A true if and only if $s|_r \Vdash A$. This does not seem unreasonable. A region r "knows" truthmaking within its area. Therefore, s makes A true according to a *world* if and only if $s \Vdash A$. So, worlds *get truthmaking right*.

Furthermore, it follows that "truthmaking is its own reward" (to use Barry Smith's words [19]). If $r \Vdash A$ then $r \Vdash E!\underline{r} \Rightarrow A$; if r makes A true, then r makes it true that r makes A true.

That deals with weak truthmaking. For strong truthmaking, we have $E!x \wedge (E!x \Rightarrow A)$, to model '$x$ makes A true'. This behaves well. We have $r \Vdash E!\underline{s} \wedge (E!\underline{s} \Rightarrow A)$ (according to r, s (successfully) makes A true) if and only if $r \Vdash E!\underline{s}$ (so $s \sqsubseteq r$) and $r \Vdash E!\underline{s} \Rightarrow A$. We have already seen that this latter fact means that $s|_r \Vdash A$. Now since $s \sqsubseteq r$, this obtains just when

$s \Vdash A$. So, according to r, s (successfully) makes A true just when s is a part of r, and s does in fact make A true.

It follows immediately, that the first of our truthmaking theses holds in a very strong sense in our models.

- $w \Vdash A \Leftrightarrow (\exists x)(E!x \wedge (E!x \Rightarrow A))$

This claim says that in any world w, A is (relevantly) equivalent to the claim that there is something which makes A true. This is our formalisation of the thesis: A is true if and only if there is something which makes A true. We can verify this thesis in the following way. We have $w \Vdash A \Leftrightarrow (\exists x)(E!x \wedge (!x \Rightarrow A))$ if and only if for every region s, $r \Vdash A$ if and only if $r \Vdash (\exists x)(E!x \wedge (E!x \Rightarrow A))$. We reason first from left to right. If $r \Vdash A$, then $r \Vdash E!\underline{r} \Rightarrow A$, and $r \Vdash E!\underline{r}$ jointly give us $r \Vdash E!\underline{r} \wedge (E!\underline{r} \Rightarrow A)$, and hence $r \Vdash (\exists x)(E!x \wedge (E!x \Rightarrow A))$ as desired. From right to left, if $r \Vdash (\exists x)(E!x \wedge (E!x \Rightarrow A))$, then there is some s where $r \Vdash E!\underline{s}$ and $r \Vdash E!\underline{s} \Rightarrow A$. It follows that $s \sqsubseteq r$, and that $s|_r \Vdash A$. But since $s \sqsubseteq r$, $s|_r = s$, so $s \Vdash A$, and finally, $r \Vdash A$ by heredity. The equivalence is given.

This verifies the first of our theses of truthmaking. The others will be verified in the same way. We will formalise then as sentences and show that these sentences hold in *all worlds*. They are necessary truths in our model.

The *conjunction thesis* is not difficult. It has two readings. First, one in terms of weak truthmaking

- $w \Vdash (\forall x)((E!x \Rightarrow A \wedge B) \Leftrightarrow (E!x \Rightarrow A) \wedge (E!x \Rightarrow B))$

and one in terms of strong truthmaking

- $w \Vdash (\forall x)(E!x \wedge (E!x \Rightarrow A \wedge B) \Leftrightarrow (E!x \wedge (E!x \Rightarrow A)) \wedge (E!x \wedge (E!x \Rightarrow B)))$

The second follows immediately from the first by distributing $E!x$ through both sides of the biconditional. The first is verified as follows. If $r \Vdash E!\underline{s} \Rightarrow A \wedge B$ if and only if $s|_r \Vdash A \wedge B$, which holds if and only if $s|_r \Vdash A$ and $s|_r \Vdash B$. This, in turn, is equivalent to $r \Vdash (E!\underline{s} \Rightarrow A) \wedge (E!\underline{s} \Rightarrow B)$. So, $w \Vdash (E!\underline{s} \Rightarrow A \wedge B) \Leftrightarrow (E!\underline{s} \Rightarrow A) \wedge (E!\underline{s} \Rightarrow B)$ for any s at all, giving us the universally quantified claim we want.

The *disjunction thesis* holds, even though entailment does not satisfy $(C \Rightarrow A \vee B) \Rightarrow (C \Rightarrow A) \vee (C \Rightarrow B)$ in general. (To see why, let $C = A \vee B$. A region in which $A \vee B$ holds need not be one in which A holds, and it need not be one in which B holds.) We *do* have $(E!\underline{r} \Rightarrow A \vee B) \Rightarrow (E!\underline{r} \Rightarrow A) \vee (E!\underline{r} \Rightarrow B)$. If $s \Vdash E!\underline{r} \Rightarrow A \vee B$ then $r|_s \Vdash A \vee B$ and hence either $r|_s \Vdash A$ or $r|_s \Vdash B$, and hence $s \Vdash E!\underline{r} \Rightarrow A$ or $s \Vdash E!\underline{r} \Rightarrow B$, and hence $s \Vdash (E!\underline{r} \Rightarrow A) \vee (E!\underline{r} \Rightarrow B)$ as desired. The converse entailment is trivial.

If $s \Vdash (E!\underline{r} \Rightarrow A) \vee (E!\underline{r} \Rightarrow B)$ then either $r|_s \Vdash A$ or $r|_s \Vdash B$, and in either case, $r|_s \Vdash A \vee B$, which gives us $s \Vdash E!\underline{r} \Rightarrow A \vee B$ as desired. So, we have the disjunction thesis in its weak truthmaking form:

- $w \Vdash (\forall x)((E!x \Rightarrow A \vee B) \Leftrightarrow (E!x \Rightarrow A) \vee (E!x \Rightarrow B))$

The strong form, given below

- $w \Vdash (\forall x)(E!x \wedge (E!x \Rightarrow A \vee B) \Leftrightarrow (E!x \wedge (E!x \Rightarrow A)) \vee (E!x \wedge (E!x \Rightarrow B)))$

follows immediately, by distributing $E!x$ through the left and and right hand expressions of the biconditional.

The *negation* clause is formalised as follows:

- $w \Vdash (\exists x)(E!x \wedge (E!x \Rightarrow {\sim}A)) \Leftrightarrow {\sim}(\exists x)(E!x \wedge (E!x \Rightarrow A))$

The restriction to strong truthmaking is necessary if we allow inconsistent regions. If r is an inconsistent region then in any world we have $E!\underline{r} \Rightarrow A$ and $E!\underline{r} \Rightarrow {\sim}A$, so the inference from 'something makes A true' to 'nothing makes ${\sim}A$ true' will fail when read as *weak* truthmaking. In the strong reading, it succeeds. We can reason as follows: $r \Vdash (\exists x)(E!x \wedge (E!x \Rightarrow {\sim}A))$ if and only if $r \Vdash {\sim}A$ by our first thesis. Similarly, $r \Vdash {\sim}(\exists x)(E!x \wedge (E!x \Rightarrow A))$ if and only if for each s where rCs, $s \nVdash (\exists x)(E!x \wedge E!x \Rightarrow A)$. That holds if and only if $s \nVdash A$ for each s where rCs. That is, $r \Vdash {\sim}(\exists x)(E!x \wedge (E!x \Rightarrow A))$ if and only if $r \Vdash {\sim}A$. So, both halves of the biconditional stand and fall in exactly the same region. Something makes ${\sim}A$ true if and only if nothing makes A true.

We will end with a discussion of the thesis analysing entailment with necessary preservation of truthmaking. In what sense is $A \Rightarrow B$ equivalent to the claim that, of necessity, every truthmaker for A is a truthmaker for B? This has a *weak* and a *strong* reading. They work differently, depending on whether we allow inconsistent regions. First, the weak reading.

- $w \Vdash (A \Rightarrow B) \Leftrightarrow \Box(\forall x)((E!x \Rightarrow A) \Rightarrow (E!x \Rightarrow B))$

This reading is uses *weak* truthmaking, and it holds unrestrictedly (whether we allow inconsistent regions or not). We can verify it as follows: $r \Vdash A \Rightarrow B$ if and only if for each s, if $s \Vdash A$ then $s|_r \Vdash B$. Now $r \Vdash \Box(\forall x)((E!x \Rightarrow A) \Rightarrow (E!x \Rightarrow B))$ if and only if $r \Vdash (E!\underline{s} \Rightarrow A) \Rightarrow \underline{s} \Rightarrow B)$ for each s (entailment statements, if true, are necessarily true). This, in turn, is true if and only if for each s and t, if $s|_t \Vdash A$ then $s|_{(t|_r)} \Vdash B$. Now, it turns out that $s|_{(t|_r)} = (s|_t)|_r$, so our condition holds if and only if for each $s|_t$, if $s|_t \Vdash A$ then $(s|_t)|_r \Vdash B$. This holds if and only if for each s, if $s \Vdash A$ then $s|_r \Vdash B$, which is the condition we have already seen for $r \Vdash A \Rightarrow B$. The equivalence holds. An entailment is true if and only if (of necessity) any (weak) truthmaker for the antecedent is a (weak) truthmaker for the consequent.

What can we do with strong truthmaking? The story here is less clear. In the case of *consistent* regions we can prove the following equivalence:

- $w \Vdash (A \Rightarrow B) \equiv \Box(\forall x)(E!x \wedge (E!x \Rightarrow A) \supset E!x \wedge (E!x \Rightarrow B))$

Now the equivalence is a *material* one. In any world, $A \Rightarrow B$ is true in that world if and only if, of necessity, any (existing) truthmaker for A is also an (existing) truthmaker for B.

We can verify the statement like this: $w \Vdash A \Rightarrow B$ if and only if for *every* region r, if $r \Vdash A$ then $r \Vdash B$. That is equivalent to the claim that $w \Vdash (E!\underline{r} \Rightarrow A) \supset (E!\underline{r} \Rightarrow B)$ for each region r. Now, we wish to restrict the truthmaking to *strong* truthmaking, by adding the claim that r *exist*. How can we do this? We can rely on the fact that each r is consistent, and therefore exists in *some* world. We have, then, that our condition is equivalent to $w' \Vdash E!\underline{r} \wedge (E!\underline{r} \Rightarrow A) \supset E!\underline{r} \wedge (E!\underline{r} \Rightarrow B)$ for each world w' and region r, but this, then, is equivalent to $w \Vdash \Box(\forall x)(E!x \wedge (E!x \Rightarrow A) \supset E!x \wedge (E!x \Rightarrow B))$ as desired. Entailment (at least with consistent regions) is necessarily equivalent to the necessary preservation of truthmaking.

Why does this reasoning fail with inconsistent regions? It fails because the entailment $A \wedge {\sim}A \Rightarrow B$ fails at worlds. There may well be a region r in which $A \wedge {\sim}A$ is true but in which B fails to be true. Now, it doesn't follow that there is some *possible* truthmaker for $A \wedge {\sim}A$ which is not a truthmaker for B. For r is inconsistent, and hence *cannot* be actual. The reduction of entailment in this context to preservation of *possible* truthmaking will not succeed. The account of "Truthmakers, Entailment and Necessity" took truthmakers to be consistent, so the current result does not call into question the theses of that paper.

This completes our journey through the theses of truthmaking. We have seen that the theses are mutually consistent, because there is a model in which all come out to be true. Hopefully the model gives some illumination of the content of the theses and perhaps it will serve as an example to help foster further analysis of the notion.

7. *Observations*

I will end this paper with a number of observations of how this model treats a number of puzzles about truthmaking.

NEGATION: This account *explicitly* defines negation in terms of relation of incompatibility. Some have thought that this is a bad thing [18]. We have defined negation in terms of a "negative" notion. Of course we have done so. There is no avoiding this, any more than we could avoid using something like conjunction in the explanation of the semantics of conjunction. However, it does not follow that incompatibility is a "part" of the truthmaker for a negative claim. There is a difference between what makes it true that $\sim A$, which might be a part of the world (a part, of course, which excludes the truth of A) and the *explanation* we give how A fails to be true. That explanation will perhaps make a reference to the relation of compatibility or incompatibility between regions or states.

MODALITIES: Truthmakers for modal claims are *regions*. As with negations, modal claims do not need *special* truthmakers. However, accessibility relations are required in the *explanation* of the truthmaking relation between regions and modal claims.

TOTALITIES: There are no explicit *totality* facts in this account, but our model does have some notion of space being *filled*. There is no sense, in this model, in which the "space" could have been different. There is no sense in which there could have been more dimensions than there actually are. This is, of course, a problem if we wish to modify the model to be *anything like* the actual world.

MORE OBJECTS: This model has very few *objects*. The only objects are *regions*. There is no sense in which we could talk of an object which is located at $\langle 5, 1\rangle$ at this world and $\langle 1, 5\rangle$ in another. This is obviously a rather thin universe. Adding more objects will make the *disjunction thesis* rather hard (perhaps *impossible*) to maintain. For an object a makes $A \vee B$ true (according to the whole world) if and only if in every region r in which a exists, $A \vee B$ is true. Now, it could well be that in some such regions r, A is true, and in others, B is true. The disjunction thesis could well fail for this kind of truthmaker. I no longer think that this is a such a bad thing. Much more must be said about this, but I will leave that for another time.[13]

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