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PRINCIPIA MATHEMATICA, PART VI: RUSSELL AND WHITEHEAD ON QUANTITY

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Abstract

The article aims at providing an introduction to Russell's and Whitehead's neglected mature theory of magnitude, presented in the last published part of *Principia Mathematica*. I intend to show that *Principia*, VI, is the culmination of a line of thought whose beginning goes back to the time of Russell's first works on the theory of relations, in 1900. But I insist as well on Whitehead's own important contribution. At the end, I address a more general problem: how to articulate this quantitative doctrine of numbers with Russell's and Whitehead's logicist stance?

1. *Introduction*

In a letter to Russell, dated 14/09/1909, Whitehead wrote:

> Dear Bertie,
>
> The importance of quantity grows upon further considerations — *The modern arithmetization of mathematics is an entire mistake* — of course a useful mistake, as turning attention upon the right points. It amounts to confining the proofs to the particular arithmetic cases whose deduction from logical premisses forms the existence theorem. But this limitation of proof leaves the whole theory of applied mathematics (measurement etc) unproved. Whereas with a true theory of quantity, analysis starts from the general idea, and the arithmetic entities fall into their place as providing the existence theorems. To consider them as the sole entities involves in fact complicated ideas by involving all sorts of irrelevancies — In short the old fashioned algebras which talked of "quantities" were right, if they had only known what "quantities" were — which they did not.

(...)

You see in short that I have recovered the simple faith of my angel infancy — I only hope that it is not a sign of decay of intellect or of approaching death — You will have to devote some attention to my MS — since their results will come as a shock to the current orthodoxy. In fact mathematicians will feel much like Scotch Presbyterians who might find that a theological professor in one of their colleges had dedicated his work to the Pope.

Yours affectionately, ANW

Russell's logicism is often seen as an extension of the arithmetization of mathematics associated with the names of Cantor and Dedekind. But according to Whitehead here, it shouldn't: "The modern arithmetization of mathematics is an entire mistake", said he — and as we can admit that, in 1909, logicism was not taken as an entire mistake by him, we can conclude that Whitehead did not view logicism as an extension of arithmetization. The following question cannot then be avoided: how to conceive logicism if not as an extension of the arithmetization's program? How to be still a logicist, while criticizing the reduction of all mathematics to arithmetic? The issue concerning the status of quantity in *Principia* has thus a central importance, since it leads immediately to raise the issue of the nature of logicism and of his relationship with arithmetization.

In this paper, my primary aim will be to flesh out what Whitehead put forward in his letter. His attack against arithmetization is not a mere bavardage: its full content is expounded in the last published part, part VI, of *Principia Mathematica*. To my knowledge, no account of this work exists in the literature.[1] Owing to the potential importance of the topic, importance brought out by Whitehead's letter, there is no doubt that this lack represents a serious defect, liable to distort our picture of logicism. My goal here will be to begin to fill this gap by offering a not too technical introduction to Russell's and Whitehead's work.

Indeed, one of the reasons why *Principia* VI has been so much neglected comes from the fact that the authors used there many complicated and esoteric notational devices, and relied heavily on the machinery presented in the first five parts. In the few pages that follow, I will not be able to do justice to the real richness of these developments. I can however facilitate the reading

[1] One finds some very brief remarks in "Whitehead and the Rise of Modern Logic" (see [Quine, 1995], p. 3–36); see as well [Quine, 1962], [Bigelow, 1988] and [Grattan-Guinness, 2000].

of the work in explaining the main ideas and in translating the basic symbolism in a more up-to-date notation. In the sequel, I will put much emphasis on the plan of *Principia* VI, and, especially, on the articulation between section A, devoted to the definition of rational and real numbers, and sections B–D, devoted to quantities and measurements.

I will first speak about the sources of *Principia*'s theory of quantity. Section 3 will focus on the definitions of rational (and real) numbers, dealt with in Part VI, A. In section 4, I will present the main lines of Russell's and Whitehead's theory of quantity and measurement (Part VI, B–D). In my last, conclusive, section, I will briefly come back to the issue concerning the relation between the doctrine of quantity and logicism taken as a whole, and also speak about Whitehead's own contribution to the theory. We know that Whitehead was in charge of *Principia* VI — does this mean that we should consider him as the sole author of this work?

2. *The sources*

Russell and Whitehead referred to Burali-Forti's paper *Les propriétés formales des opérations algébriques* as a source for their theory.[2] In this work, Burali-Forti attempted to axiomatize Euclid's theory of quantity. He was not, at the time, the only one pursuing such a project — at the turning of the Century, many mathematicians offered various formal theory of quantity.[3] Burali-Forti's program was peculiar, however, in that he explicitly aimed to oppose the arithmetization view, according to which continuous magnitudes can be reduced to the whole numbers. Burali-Forti thus explained:[4]

> Chapter I of this book contains the properties of the magnitudes which do not depend on the idea of number (integer or fraction or irrational). (...) Chapter II contains the basis of the theory of the

[2] Whitehead alludes to this work in a letter to Russell dated 28/1/1913: "As to the preface — The work on 'grandeurs' started with a study of Burali-Forti's articles in the *Rivista* and was directed initially to arrive at the same results. Of course his work is really based on Euclid Bk V — whom I ought also to have studied, but did not. Thus our antecedents are Euclid and Burali-Forti; but it should be mentioned that (1) by the introduction of 'relations' and (2) by the keeping of the group idea in the background, and (3) by the separate treatment of ratio, and (4) by avoiding number and (5) by the introduction of cyclic groups, the subject has been entirely modified. I think these points should be mentioned somewhere, not to claim novelty, but to show people what to look for".

[3] For more on this, see [Gandon, 2008b].

[4] [Burali-Forti, 1899], p. 34.

> whole numbers. The idea of a whole number is logically derived from the usual and concrete idea of magnitude. (...)
>
> An analogous procedure is followed in chapters III and IV, devoted to the rationals and the irrationals.
>
> A first conclusion of the method we just exposed is the swiftness with which we can teach (...) the formal properties of the algebraic operations, by including the elements magnitudes and numbers which are usually examined separately. But another much more important conclusion is reached, which consists in making it possible to obtain the general idea of number in a concrete shape by deriving it from the concrete usual idea of magnitude, which is essential for the metrical part of geometry as well.

While Dedekind was proud to define the real numbers without resorting to the idea of continuous quantity,[5] Burali-Forti attempted, in a complete reversal which is as well a genuine return to the Euclidean tradition, to base arithmetic and real analysis on the concept of quantity.

I list the axioms (translated in today's usual notation) which defined a set G of homogeneous magnitudes $\langle G, +\rangle$ in [Burali-Forti, 1899]:

1- $a, b \in G, a + b = b + a$
2- $a, b \in G, a + (b + c) = (a + b) + c = a + b + c$
3- $a, b, c \in G, (a + c = b + c) \Rightarrow a = b$
Definition of order: if $a, b \in G$ then $a > b$ iff $\exists x \in G\backslash\{0\}, a = b + x$
4′- $a \in G, \exists x \in G, a + x = a$
4″- $a \in G, \exists x \in G, a + x > a$
5- $b \in G, a \in G\backslash\{0\}, (a + b) \in G\backslash\{0\}$ [6]
6- $a, b \in G, a = b \vee a < b \vee a > b$
7- $a \in G, \exists x \in G\backslash\{0\}, x < a$
8- If $U \subset G, U \neq \varnothing, \exists x \in G, \forall y \in U, y < x$, then:
$\exists z \in G, \forall v \in G(v < z \Leftrightarrow \exists w \in U, v < w)$.

[5] See the preface of [Dedekind, 1888]: "All the more beautiful it appears to me that without any notion of measurable quantities and simply by a finite system of simple thought-steps man can advance to the creation of the pure continuous number-domain; and only by this means in my view is it possible for him to render the notion of continuous space clear and definite."

[6] Axiom 5 implies that the semi-group $\langle G, +\rangle$ is strictly positively ordered.

In modern terms, Burali-Forti defined a dense and complete[7] monoïd.[8] The numbers were next introduced as ratios of homogeneous quantities.[9] In the rational case, which I will focus on because it is simpler than the real case, two *grandeurs* A and B had the ratio m/n iff $nB = mA$. The ratio m/n was said to be greater than another p/q when $mq > np$. From axiom 7 (which insures that $\langle G, > \rangle$ is dense), one could easily derive the density of the ordered set of the ratios. In a similar way, Burali-Forti proved the completeness of his ordered set of real ratios from axiom 8 (the Dedekindian condition).

Before having been the source of 1913's theory of magnitude, Burali-Forti's construction was at the basis of Russell's doctrine of distance, first developed in the English version of the seminal *On the Logic of Relations*,[10] and resumed in the *The Principles*.[11] As this theory, simpler than the mature one, set the stage for *Principia*'s doctrine, I will briefly expound it.

[7] Axiom 8 is a version of the Dedekindian condition. Burali-Forti proves that the Archmedian hypothesis follows from the axioms.

[8] A monoïd $\langle G, + \rangle$ is a set endowed with an associative binary operation which contains a neutral element. More generally, at the end of the XIXth Century, one always found two structures in the theories of quantity: a group or a semigroup structure (which gives sense to the addition between magnitudes) and an ordinal structure (which gives sense to the idea that a quantity can be greater than another).

[9] The ratios are defined by the standard equimultiple condition:

$$\forall a, b \in G, \forall c, d \in G\backslash\{0\}\ (a : b = c : d \\ \iff \forall m, n \in \mathbb{N}\ (ma <=> nb \iff mc <=> nd))$$

[10] In the published version, the sections devoted to quantities have been eliminated; compare [Russell, 1900] and [Russell, 1901].

[11] A copy of Burali-Forti's article, annotated by Russell, can be found in the Russell Archives (McMaster University), and many details in the construction evoke Burali-Forti's approach. There is however one important difference between the two theories. The Italian mathematician starts with only one indefinable, the additive operation — order is derived (see section 2). For Russell, on the other hand, order is a primitive concept. If the difference does not greatly change the shape of the formal structure, it deeply affects the general conception of magnitude. Indeed, for Russell, magnitude in general is primarily defined by a "capacity for the relation of *greater* and *less*", not by a capacity for divisibility; see [Russell, 1903] p. 159. For more on the role of order in Russell's theory, see [Michell, 1999].

Here is the definition of a kind of distance Δ given by Russell:[12]

$$\Delta =_{\text{Df}} FG \cap L \ni \{x, y \epsilon \lambda . \supset_{x,y} . \exists L \cap R \ni (xRy) : Q = R_L .$$
$$R_1, R_2, R_3 \epsilon L \ . \ R_1 Q R_2 \ . \supset_{R_1, R_2, R_3} . \ R_1 R_2 = R_2 R_1 \ . \ R_1 R_3 Q R_2 R_3\}$$

Russell comments:[13]

> This is a definition of a kind of distance, *i.e.* of a class of distances which are quantitatively comparable. A kind of distance is a series in which there is a term between any two, and it is also a group.

Thus, despite the strangeness of the notation, distance (or magnitude) is here defined in a very standard way, as an ordered series (a F), which is also a group (a G). Some further continuity conditions are then added, which make Russell's notion very similar to Burali-Forti's *grandeurs homogènes* — indeed, Russell's distance is isomorphic to $\langle \mathbb{R}, + \rangle$, while Burali-Forti's magnitude is isomorphic to $\langle \mathbb{R}^+, + \rangle$.

However, the true import of the Russellian approach lies elsewhere, in the way the group structure is conceived. In another passage of the same manuscript ([Russell, 1900] p. 594), the notion is introduced that way:

$$G =_{\text{Df}} Cls' 1 \to 1 \cap K$$
$$\ni \{P \epsilon K . \supset_P . \breve{P} \epsilon K : P, R \epsilon K . \supset_{P,R} . PR \epsilon K . \pi = \rho\}$$

That is, a group is a set K of one-one relations onto defined over the same field $\pi = \rho$ (in other words: K is a class of permutations of a given set) such that, firstly, if P belongs to K, the converse $\breve{P}$ belongs to K, and such that, secondly, if P and R belong to K, the relative product PR belongs to K. In other words, Russell defines a group as a permutation group defined on an underlying set.[14] Now, Cayley's theorem, well-known by Russell and all the

[12] F designates a dense series, G a group. L is a distance or a magnitude of a specific kind, *i.e.* as Burali-Forti's calls it, an homogenous magnitude. Q is the order relation of the magnitude ($Q = R_L$), and λ is the field of the magnitude (*i.e.* the field of the group and of the series). $R_1, R_2, R_3 \epsilon L \ . \supset_{R_1, R_2} . \ R_1 R_2$ means that the group is commutative. $R_1, R_2, R_3 \epsilon L \ . \ R_1 Q R_2 \ . \supset_{R_1, R_2, R_3} . \ R_1 R_3 Q R_2 R_3$ means that the ordinal relations and group operation are compatible.

[13] [Russell, 1900], p. 609.

[14] In group theory, a permutation of a set S is any bijective function taking S onto S; a permutation group is a group whose elements are permutations of a given set S, and whose group operation is the composition of permutations.

mathematicians of the beginning of the XXth Century, assures us that every abstract group G is isomorphic to a permutation group. No loss of generality is thus caused by Russell's definition of a group in terms of permutation group, and Russell knew very well he had the right to proceed that way. But he also knew that he could have characterised the group structure in (what seems to be) a more straightforward way, by just setting the standard axioms. Why did he not do that? Why did he take a path which seems to be more complicated than the standard abstract one?

From Russell's standpoint, the group operation posed a problem: indeed, how to account, in the new logic of relations, for addition? Is it a three-terms relation, a combination of a relation with identity, or a new kind of term?[15] Russell's answer in 1900 was to say that a group operation is a relative product (a relation of relations) defined on a special set of relations — worded in a more contemporary terminology: addition was then conceived as a composition between bijective mappings. This solved the difficulty about the nature of the group operation, and allowed Russell to take the theory of group as a part of his new logic of relations. The elements of the group were relations, and the group operation was also a relation.

This constitutes a first change with respect to Burali-Forti's approach. For Russell, quantities were not the elements of an abstract structure defined axiomatically, but some relations belonging to a certain set; accordingly, additivity was not any operation which has the required properties, but a composition of relations. There is however another, more general, disagreement: Russell did not adhere to Burali-Forti's anti-arithmetization program. Indeed, in the rest of [Russell, 1900], as in [Russell, 1903], Russell resumed Cantor's and Dedekind's definitions of $\mathbb{R}$; his own favorite characterisation was a slightly amended version of Dedekind's notorious cut definition.[16] If Russell found in [Burali-Forti, 1899] his technical inspiration, he thus did not share the philosophical background of the work.

Be that as it may, the idea that quantities are relations and that quantitative addition is a relative product remained the hallmark of Russell's and Whitehead's approach to quantity in *Principia Mathematica*: the vector families (the 1913 word for the notion of kind of magnitudes) were defined as some sets of relations and the addition between quantities was still thought

[15] For a discussion of this problem, see [Sackur, 2005] pp. 143–209.

[16] See [Russell, 1903], Part V.

in terms of relative product.[17] There were however two major changes in the doctrine brought forward in *Principia* VI:

1) As we have seen, in 1900 (and in 1903), the real and the rational numbers were defined, in a purely Dedekindian way, without any reference to quantity; in 1913, Russell's and Whitehead's definition linked the numbers to their application in the measurement of quantities. It seems then that, as time goes by, a rapprochement with Burali-Forti occurred. As we will see, things are much more complicated, however.

2) In 1900 and 1903, the quantitative structure of distance was very rich, since it was isomorphic to $\langle \mathbb{R}, + \rangle$; this, of course, excluded from consideration many kinds of quantitative domains. In *Principia*, the definition of magnitude was considerably generalized, and could accommodate many different quantitative structures.

I will examine each of these changes in turn.

3. *Numbers and magnitudes in* Principia Mathematica

The first thing to note is that generalization of numbers comes very late — this is only in the first section of *Principia* VI that the negative, the rational and the real numbers are introduced. That is, more than two third of the work is written without resorting to any notion of numbers (beyond the integers). In particular, the whole theory of series is developed in part V. This means that, in *Principia*, real analysis is constructed without any reference to reals. Better, Russell and Whitehead explicitly contend that one of the advantage of their presentation is to show that real analysis, in its essence, has nothing to do with numbers and measurement:[18]

[17] In the summary of part *303 devoted to ratios, we even find an attempt to justify this move [Russell and Whitehead, 1913], p. 260–261): "This definition [of ratios as relation of relations] requires justification [...]: we commonly think of ratios as applying to magnitudes other than relations. [...] In applying our theory to (say) the ratio of two masses, we note that the idea of quantity (say, of mass) in any usage depends upon a comparison of different quantities. The "vector quantity" R, which relates a quantity m_1 with a quantity m_2, is the relation arising from the existence of some definite physical process of addition by which a body of mass m_1 will be transformed into another body of mass m_2. Thus σ steps, symbolized by R^σ, represents the addition of the mass $\sigma(m_2 - m_1)$. [...] Thus to say that an entity possesses μ units of quantity means that, taking U to represent the unit vector quantity, U^μ relates the zero of quantity — whatever that mean in reference to that kind of quantity — with the quantity possessed by that entity. It can be claimed for this method of symbolizing the ideas of quantity (α) that it is always a possible method of procedure whatever view be taken of it as a representation of first principles, and (β) that it directly represents the principle "No *quantity* of any kind without a comparison of different quantities of that kind." "

[18] [Russell and Whitehead, 1912], pp. 687. On this point, Russell is certainly indebted to Hausdorff, who, in [Hausdorff, 1906], developed a pure and general theory of order types.

> In the definitions usually given in treatises on analysis, it is assumed that both the arguments and the values of the functions are numbers of some kind, generally real numbers, and limits are taken with respect to the order of magnitudes. There is, however, nothing essential in the definitions to demand so narrow a hypothesis. What is essential is that the arguments should be given as belonging to a series, and that the values should also be given as belonging to a series, which need not be the same series as that to which the arguments belong. In what follows, therefore, we assume that all the possible arguments to our function, or at any rate all the arguments which we consider, belong to the field of a certain relation Q, which, in cases where our definitions are useful, will be a serial relation.

Think for instance of Russell's and Whitehead's discussions of the temporal series;[19] both philosophers directly used the concepts of convergence and limit to investigate the nature of the temporal continuum, without first correlating the series of instants to the series of numbers. This is in line with *Principia*. For Russell and Whitehead, real analysis could immediately be applied to non-numerical series — from their point of view, any detour by numbers and measurements appeared as an artificial complication, merely caused by the fact that the analysis of the fundamental notions of calculus has not been generalized enough.

Now, if we do not need the reals for developing mathematical analysis, why then introduce them? Russell and Whitehead claimed, at the beginning of part VI, that generalization of numbers is necessary as soon as we want to account for measurement: "The purpose of this Part [VI] is to explain the kinds of applications of numbers which may be called *measurement* [and] for this purpose, we have first to consider generalizations of number" ([Russell and Whitehead, 1913], p. 233). Thus, the logicists clearly endorsed the principle (sometimes, called application constraint[20]) according to which the applicability of reals and rationals to measurement should be built into their very definitions.

The idea is expounded in Part VI section A. For our purpose, it will be sufficient to focus only on the account of positive ratios — we will leave aside the definition of the reals, and all the difficulties involved by the questions of

[19] [Russell, 1914], and [Whitehead, 1920].

[20] See [Wright, 2000], who also uses the term "Frege's Constraint".

types (that is, we will accept the axiom of infinity[21]). How did Russell and Whitehead define the rational numbers?

In Part VI of *Principia*, the inductive cardinals, and the operations of addition and multiplication between them have already been defined. Russell and Whitehead use these notions to define the relation Prm between two couples of whole numbers (ρ, σ) and (μ, ν): $(\rho, \sigma)\mathrm{Prm}(\mu, \nu)$ iff ρ is prime relative to σ and $\rho \times \nu = \sigma \times \mu$ (that is, iff ρ/σ is the irreducible ratio equal to μ/ν). The ratio μ/ν between two integers ($\nu \neq 0$) is now defined in the following way (*303.01, *ibid.*, p. 260):[22]

$$\mu/\nu =_{\mathrm{Df}} \hat{R}\hat{S}\{(\exists\rho,\sigma).(\rho,\sigma)\mathrm{Prm}(\mu,\nu).\dot{\exists}!R^{\sigma}\dot{\cap}S^{\rho}\}$$

In other words, a ratio μ/ν is a relation between two binary relations R and S (Russell and Whitehead write $R(\mu/\nu)S$), such that:

(R1) $\exists\rho, \sigma \in \mathbb{N}, (\rho, \sigma)\mathrm{Prm}(\mu, \nu)$

(R2) There are two objects x and y such that the relation R^{σ} and S^{ρ} hold between them.[23]

Two remarks about this definition:

1) The second condition has an obvious resemblance with Euclid's definition of ratio, resumed by Burali-Forti. Imagine that R is a certain positive distance between two points on the line, and that S is another such a distance; then Russell and Whitehead say that R and S have the ratio μ/ν, if one can find a point from which ν steps of size R makes one reach exactly

[21] As the authors explained ([Russell and Whitehead, 1913], p. 234): "Great difficulties are caused, in this section, by the existence-theorems and the question of types. These difficulties disappear if the axiom of infinity is assumed, but it seems improper to make the theory of (say) 2/3 depend upon the assumption that the number of objects in the universe is not finite. We have, accordingly, taken pains not to make this assumption, except where, as in the theory of real numbers, it is really essential, and not merely convenient."

[22] Here, the type constraints are left implicit. But of course, in the following definition, if x and y are object variables, R and S must be relations which take objects as arguments.

[23] $xR^{\sigma}y$ means $x\underbrace{R \mid R \mid R \mid ...R}_{\sigma \text{ times}}y$, where "|" denotes the "relational product" — or the composition operation between binary relations (recall that if R and S are two binary relations such that $R \subset X \times Y$ and $S \subset U \times Z$, then $S \mid R = \{(x, z) \in X \times Z : \exists y \in (Y \cap U), (x, y) \in R \wedge (y, z) \in S\}$).

the same point as μ steps of size S.[24] Thus, (R2) is a first move toward explaining how rational numbers can be used for measuring quantities.

2) (R2) did not play any role in the derivation of the arithmetical (algebraic and order-theoretic) properties of ratios. From a mathematical perspective, the condition $\rho \times \nu = \sigma \times \mu$, given in (R1), is the mainspring of the construction — this is not surprising, owing that it is the same condition as the one used for defining the equivalence relation between couples of integers in the standard definition of the rational numbers. From (R1), and from the definitions of order and of the operations between ratios, Russell and Whitehead derive all the usual algebraic and ordinal properties of $\mathbb{Q}^+$, without the slightest difficulty (the questions of type left aside). The density of $\mathbb{Q}^+$, for instance, directly derives from (R1) and from the definitions of order and addition, and the idea that rationals are relations of relations do not play any role in the deduction.[25] In particular, no assumptions on the domain of relations to which the ratios are applied is required in the derivation of the standard mathematical properties of the rationals. From a purely arithmetical point of view, the relational part of *303. 01 is completely idle.

To summarize, 1913's theory of rationals exhibits two very important features: first, a connection is set between numbers and relations: rationals are now said to be relations of relations; second, this new definition does not compel Russell and Whitehead to put some special formal constraints on the fields of relations the rationals are applied to.

The first point shows that Russell and Whitehead endorsed a version of the application constraint. But the second point seems to limit the scope of the first move. Let me explain why by comparing their approach to Burali-Forti's one. Burali-Forti adhered to the application constraint, and defined a real number as a ratio between two members of G. But for him, the arithmetical properties of the ratios were inherited from the formal features of

[24] Note, however, that (R2) is an existential claim. This means that $R(\mu/\nu)S$ does not forbid that one can find another point on the line and another ratio γ/δ, such that δ steps of size R from this point lead exactly where γ steps of size S lead. That is, nothing, in the account given so far, excludes the possibility that two relations have more than one ratio (see [Russell and Whitehead, 1913], p. 261–262). This fact plays an important role, as it will soon be clear.

[25] I cannot enter into too much details here, but the sole effect of (R2) is to increase the difficulty of the derivations of the usual arithmetical properties of the ratios. For instance, r and p being two rationals, in order to secure that $(r + p)/2$ is different from the empty relations, Russell and Whitehead have to show that there are two relations R and S such that $R[(r + p)/2]S$. That is, they have to show that there are sufficiently many distinct relations to instantiate all the ratios. That such is the case, is a consequence of the axiom of infinity.

G to such an extent that any weakening in the definition of the quantitative domain was forbidden. For instance, the removal of axioms 7 and 8 would make us loose the density and completeness of the set of ratios. The arithmetical properties of Burali-Forti's ratios depended then on the formal features of the fields the ratios were applied to. Such is not the case in *Principia*. Rational (and real) numbers are here as well characterized as ratios (relations of relations) — but the expected mathematical properties of the *definienda* are derived without setting any constraints on the field the numbers are applied to. The link tied between numbers and quantities seems thus less tight in *Principia* than in *Les propriétés formales des opérations algébriques*. The difference comes, of course, from the fact that ratios are, in *Principia*, relations of relations, and not relations of magnitudes. In order to better understand this difference, we will now turn on the doctrine of magnitude, presented in section B of *Principia* VI, and on the theory of measurement, studied in sections C–D.

4. *The concept of vector family*

Russell and Whitehead begin section B by defining the notion of a correspondence Φ on a common domain α. Such a correspondence is the semi-group[26] $\langle \Phi, + \rangle$ of the injective mappings on α.[27] That is:

$$\Phi = \{f : \alpha \rightarrow \alpha \mid f \text{is an injective mapping}\}$$

Now, a vector-family κ defined on α is introduced as follow:

1- κ is not empty and $\kappa \subseteq \Phi$
2- $\forall f \forall g \in \kappa, f.g = g.f$

The notion of vector family replaces the former concept of a kind of magnitude. It should then be compared to the old notion of a kind of distance, or to Burali-Forti's homogeneous magnitude. Now, what emerges from such comparing is that the vector family κ has a very weak structure. It has lost nearly all its algebraic content — κ is not a symmetrical structure (each element has not necessarily an inverse); the identity does not necessarily belong to it; worse, κ is not necessarily closed under the composition operation.

[26] A semi-group $\langle G, + \rangle$ is a set endowed with an associative binary operation.

[27] A representation theorem about semi-groups (a kind of extension of Cayley's theorem) says that every abstract semi-group can be represented as a semi-group of injective mappings; for more on this, see [Ljapin, 1960].

Among the algebraic properties, only the commutativity condition is saved. Of the rich ordinal structure of the distance, nothing remains.[28] One thing stood the test of time, however: even if it is no longer defined as a permutation group, quantity is still regarded in 1913 as a structure which operates over a set. The concept of magnitude is much more general than it was before; but it is still defined as a family of relations which acts on a common domain α.

The main interest of this blunt generalization is to allow the definition of many distinct sorts of quantity. The former notion of a kind of distance is, of course, still considered as a vector family — but it is a very special sort of quantitative structure, which, besides, does not receive much attention in *Principia*.

The "connected" families are the most important kind of vector family (κ is connected if it has at least one connected point, that is, if there is "one member of α from which we can reach any member of α by a vector belonging to the family or by the converse of a vector belonging to the family" (*Ibid.*, p. 341)). Other kinds of species are also defined: the initial families,[29] the open families,[30] the cyclic families,[31] the serial families,[32] the families in which a series of vectors can be defined (see *336) and the submultipliable[33] families. I cannot enter into the details of Whitehead's construction here, but this enumeration suffices, I hope, to give an idea of how various the directions in which the basic weak structure is developed are. Russell and Whitehead set some refined conditions on the inverse operation and on

[28] This last point moves the mature theory very far away from Russell's former doctrine, where quantity was an ordinal concept.

[29] In which there is a point in the field α which is a starting point but not an end-point of non-zero vectors; *ibid.* p. 390.

[30] "In which no number of repetitions of a non-zero member of κ_ι will bring us back to our starting point". Russell and Whitehead define the new relational structure κ_ι as the set of relations such that, if $g, h \in \kappa$, then $g^{-1}.h \in \kappa_\iota$. The relation of κ_ι are not necessarily defined on the whole of α.

[31] The definition, too complicated to be given here, is exposed in *Ibid.*, p. 458.

[32] In which every points are connected, and in which every points are such that, if $g, h \in \kappa$, then there is a $t \in \kappa$ such that $g.h(x) = t(x)$; see *Ibid.*, p. 385.

[33] "One in which any vector can be divided into ν equal parts (where ν is any inductive cardinal other than 0)"; *Ibid.*, p. 418.

the closure of the relative product; they introduce some ordinal relations between elements of α and/or of κ; they also put some constraints on the way κ acts over α.

Now that we have briefly seen how general is the notion of quantity, let me focus on the concept of measurement. In *Principia*, numbers and quantities are independent from each other. As we have seen, the arithmetical properties of numbers, derived in section A, do not depend in any way on the shape of any given vector family. And vice-versa, the quantitative structures, examined in section B, do not have to be measurable. Rational (real) numbers and vector families are described and studied for their own sake, independently from each other. In particular, Russell and Whitehead underline that a couple of relations being given, there can be several distinct ratios which relate them (see footnote 24), so that, at the beginning of section C, it is impossible to consider a ratio as a measure of a vector.

In sections C–D of Part VI, Russell and Whitehead explain that, if some appropriate restrictions are set on the vector families, then numbers can be regarded as measures of vectors. What it is, then, for a vector family, to be measurable?

This question is not an easy one. I simplify here an analysis which is, in its detail, much more complicated.[34] Russell and Whitehead give in section C a series of four conditions which can be used when the family to be measured is open:

> (1) No two members of a family must have two different ratios. [...]
> (2) All ratios [except 0 and ∞] must be one-one relations when limited to a single family. [...]
> (3) The relative product of two applied ratios ought to be equal to the arithmetical product of the corresponding pure ratios with its field limited [...]. That is to say 'two-thirds of half a pound of cheese ought to be $(2/3 \times 1/2)$ of a pound of cheese; and similarly in any other case. [...]
> (4) If X, Y are ratios, and [if R, S and T are members of the family κ such that RXT and SYT], we ought to have $[[R \mid S](X+Y)T]$, that is two-thirds of a pound of cheese together with half a pound of cheese ought to be $(2/3 + 1/2)$ of a pound of cheese, and similarly in any other instance.[35]

[34] In section D, Russell and Whitehead offer another definition of measurement, which accounts for the measure of angles. I leave this part aside here.

[35] The four conditions, written in a more modern form, concern the relation between the powers of the relation in the set:
a) $\forall g, h \in \kappa$, if $\exists n, m \in \mathbb{N}^*$ such as $g^n = h^m$, then $\nexists n, m \in \mathbb{N}^*, n/m \neq r/s$ and $g^r = h^s$.

The two first constraints aim at insuring the 'regularity' of the association of vectors to numbers — two vectors should not have more than one ratio, and, for each ratio, there should be one and only one vector which has this ratio to a given relation. The last two conditions are intended for securing a compatibility between the usual arithmetical operations and the relative product. That is, if $A_T(R)$ designates the[36] ratio p such that $R(p)T$ (that is, the measure of R according to the unity T), the two last demands say that:

(1) $A_T(R) = A_S(R) \times A_T(S)$, or the measure of R according to the unity T is equal to the measure of R according to the unity S multiplied by the measure of S according to the unity T.

(2) $A_S(T \mid R) = A_S(T) + A_S(R)$, or the measure of the relative product of T and R according to the unity S, is equal to the sum of the measures of T and of R, each of them taken according to the unity S.

Russell and Whitehead show that the open connected submultipliable vector families[37] are measurable in this sense. This means that if the rationals (the reals), defined as they are defined in section A, are restricted to this kind of family, then, a unity being chosen, the rationals (the reals) can be regarded as measures of the relations of the family, in the sense that there is a one-one correlation between a given set of rational (or real) numbers and the vectors of the family, and that this correlation satisfy the two conditions (1), (2) set out above. Russell and Whitehead call "applied numbers" the numbers whose fields are restricted to a vector family — and they call "pure numbers" the ratios and reals which are not subjected to this restriction. Then, applied numbers, when they are restricted to a measurable family, can be taken as measures of some magnitudes. On the other hand, pure numbers, even if they are applied to relations (since they are relations of relations), cannot be regarded as measures of quantity.

The framework presented in *Principia* VI is then quite subtle. Unlike Burali-Forti, Russell and Whitehead do not define pure numbers as ratios of quantities. They do not, however, content themselves to resume Dedekind's or Cantor's definition of rationals and reals in terms of certain complicated sets (of sets) of whole numbers. Instead, they define numbers as relations of

b) If $\exists k, l \in \kappa$ such as $\exists m, n \in \mathbb{N}^*, k^m = l^n$, then $\forall f \in \kappa, \exists g \in \kappa$ such as $g^n = f^m$.
If $\exists k, l \in \kappa$ such as $\exists m, n \in \mathbb{N}^*$, then $\forall f, g, h \in \kappa$, if $f^n = g^m = h^m$, then $g = h$.
c) $\exists n, m, r, s \in \mathbb{N}^*, \forall f, g, h \in \kappa, f^n = g^m$ and $g^r = h^s$ if and only if $f^{nr} = h^{ms}$.
d) $\forall f, g, h \in \kappa, \exists m, nr, s \in \mathbb{N}^*, f^m = h^n$ and $g^r = h^s$
if and only if $(f.g \in \kappa \Rightarrow (f.g)^{m+r} = h^{ns})$.

[36] The first two conditions insure that this ratio is unique.

[37] There is another condition that I leave aside.

relations and, in so doing, they preserve the resemblance with Euclid's definition (if relations are interpreted as magnitudes, then the definition is very close to the traditional one). At the same time, they cut the link between the structure of the rational (and real) ordered field and the formal properties of the quantitative domains. Russell's and Whitehead's pure numbers get their arithmetical properties directly from their definition in terms of relations of relations, not from some special formal features of the structure of the relations they are applied to.

In this construction, the key idea is to define the quantities not as the elements belonging to a particular axiomatically defined structure (as in Burali-Forti's homogeneous magnitudes), but as some relations. This move allows Russell and Whitehead to consider the relationship between pure and applied numbers as a relation of particularization: applied ratios are just pure ratios, restricted to a certain domain. The nature of the link tied between numbers and measurement (in other words, between pure and applied numbers) sets the logicists apart from both the arithmetizers and Burali-Forti. From the former, *Principia* VI keeps the idea that the arithmetical properties of numbers (pure numbers) should not be made dependent upon the shape of any quantitative domain; from the latter, *Principia* VI retains the idea that a good definition of numbers should explain how they are applied. Russell and Whitehead open then a new path, which, while avoiding the complete separation between numbers and measurement that we found in Cantor and Dedekind, does not follow Burali-Forti's way of reasoning. In *Principia*, as in the Euclidean tradition, numbers and magnitudes are made for each other. But this connection does not mean that the properties of numbers are inherited from the properties of the quantitative domains. On this point, the logicists remain faithful to the arithmetizer's demand.

5. *Whitehead, logicism, and the Northwest Passage*

The (brief) account I have just sketched fleshes out, I hope, the rough picture drawn in Whitehead's letter. *Principia* VI is indeed an attempt to account for "the whole theory of applied mathematics (measurement etc)" that arithmetization left "unproved". Having read this part, one can understand as well why "the old fashioned algebras which talked of "quantities" were right, if they had only known what "quantities" were" (namely, relations). Whitehead's talk is thus not empty: it describes a program which is fully carried out in *Principia*.

The most interesting ingredient of this doctrine is the subtle balance it sets up between the demand of the application constraint and the wish to insure the logical purity of arithmetical truths. This feature could be glossed along very different lines. I have chosen, here, to focus on the relationship between

Principia VI and the logicist program — and to focus also on the question whether the theory of quantity should be attributed to Whitehead alone, or to Russell and Whitehead. As I will explain now, these two issues are, in fact, related.

In the literature, the following story is often told:[38] Weierstrass, Cantor and Dedekind constructed, each in their own way, the real field and real analysis from elementary arithmetic; Russell, after Frege, added his contribution to the construction, in defining whole numbers as sets of equinumerous sets and in founding therefore arithmetic on set theory. According to Russell, we would have not only this schema:

Real analysis and real numbers
Arithmetic and whole numbers

but also this stronger one:

Real analysis and real numbers
Arithmetic and whole numbers
Logic and set theory

The 'arithmetizers' succeeded in going from the second to the third level; Russell (after Frege) would have pushed the reduction one step further. Such an account is made by Russell himself at the beginning of his *Introduction to Mathematical Philosophy* (p. 5):

> Having reduced all traditional pure mathematics to the theory of the natural numbers, the next step in logical analysis was to reduce this theory itself to the smallest set of premisses and undefined terms from which it could be derived.

As we have seen, however, in *Principia* VI, Russell and Whitehead did challenge Cantor's and Dedekind's constructions for having ignored the application constraint. And the disagreement runs very deep, since Euclid's traditional approach, the one that the arithmetizers were opposed to, is explicitly acknowledged as the main source of inspiration. How to reconcile Russell's description of logicism as an extension of the works of Weierstrass, Cantor and Dedekind with the content of *Principia* VI?

One possible answer would be to refer this apparent tension to a difference between Russell's and Whitehead's conception. *Principia* VI has been

[38] For more on this, see [Gandon 2008b].

written by Whitehead, not by Russell. Couldn't we then attribute the anti-arithmetization stance only to Whitehead, and find in this way a means to dispel the seeming discrepancy between logicism and theory of magnitude? The idea would be to acknowledge that *Principia* VI is at odds with the main line of *Principia*, but to explain the divergence by the fact that Russell wrote the first five parts and Whitehead the sixth one.

This hypothesis does not stand up to scrutiny, however. Recall first that Russell developed a relational theory of quantity as early as 1900. Section B of part VI is nothing but a generalization of this early work. So Whitehead then did not start from nothing — he heavily leaned on Russell's own previous writing. It is thus difficult to completely sever his contribution with Russell's works. What is more, Russell and Whitehead sent their manuscripts to each other; they discussed them in their abundant correspondence, and during numerous regular meetings. So, even if Whitehead was the driving force behind part VI, Russell knew and agreed with what he was doing. Thus, in a letter to Jourdain, dated 21/3/1910, Russell, after having recognised that one could define, in a standard way, a ratio as a relation between integers, added ([Grattan-Guinness, 1977], p. 130):

> I have now accepted from Whitehead a new quantitative (non-arithmetical) definition of μ/ν, according to which it is a relation of vectors R, S which holds (broadly) whenever $\dot{\exists}!R^{\nu}\dot{\cap}S^{\mu}$. This enables you to take two-thirds of a pound of butter without an elaborate arithmetical detour.

The initial "I have now accepted from Whitehead" is clear: *Principia*'s theory of number and quantity has been devised by Whitehead, but understood and accepted by Russell.

We cannot hope, then, to reconcile Russell's presentation of logicism to the content of *Principia* VI by exploiting the fact that the book was coauthored. The doctrine of quantity shows that the logicist program cannot be seen as a mere extension of arithmetization — and this is precisely what makes the theory of magnitude so interesting. According to the often repeated story, logicism would be a kind of Northwest Passage project: the main and sole problem of Russell would be to go from logic and set-theory (the Atlantic ocean, let say) to arithmetic (the Pacific ocean); between the two, a very rough and difficult area, mined by the logical and set-theoretical paradoxes (the North of Canada), would have to be crossed, and a channel across the hostile land to be discovered. In other words, the logicist challenge is often construed as an existence-problem: the difficulty is to find at least one path. If it ever turned out that many such passages existed, it would be very good news, but the existence of one channel is sufficient to satisfy the logicists.

Now, I do not want to undermine the importance of the existence-problem. My claim is, however, that the Northwest Passage issue is only a part of the story. When one reads the mathematical parts of *The Principles* or of *Principia*, that is, the parts which are situated at the middle or near the end of the books (and which are usually completely neglected), one immediately sees that, most of the time, the problems Russell (and Whitehead) faced were related to the surplus of possible ways to define a mathematical theory. The issue was thus not to find a passage, but to pick up one among many available ones. For instance, in 1903, Russell had before him many possible logical definitions of a projective space. He chose one, and tried at length to explain his choice.[39] The same holds for the definition of order in part IV of [Russell, 1903]: not less than six different definitions of order were presented and discussed in chapter 24. In many cases, thus, Russell's and Whitehead's problem was not an existence, but a uniqueness issue. The task was not to secure a reduction — it was to decide which to choose among the many possible ones.[40]

Principia VI is a spectacular example of this line of thought. Russell's theory of relations was powerful enough to encompass Dedekind's definition of the real numbers. But this did not mean that Russell should resume it. Many others approaches were possible, and the issue was to decide which one among them should be taken. Now, Dedekind's theory did not explain how numbers could be applied to quantity — and, from Russell's and Whitehead's point of view, the application constraint seemed to be a sensible demand. *Principia* VI is just the demonstration that the new logic could account for the mathematical structure of the rational and the real numbers, while fulfilling the application constraint. The logicist program was not given up – the properties of the rational and real numbers were still presented as logical properties (and on this point, Russell and Whitehead did not follow Burali-Forti). But the arithmetization was clearly abandoned: the structure of $\mathbb{Q}$ and $\mathbb{R}$ was not founded on the sole structure of $\mathbb{N}$.

In order to understand how Whitehead's theory of quantity is not incompatible to the logicist program, one has therefore to renounce the Northwest Passage view — or, at least, to relativize it to the beginning of the logicist venture. For Russell, it is true, there are not many ways to reach arithmetic

[39] On this point, see [Gandon, 2008a].

[40] On this problem of multiple possible reductions, see [Benacerraf, 1965]. Note that Benacerraf regards his argument as directed against Russell's logicism. According to me, Benacerraf's attack misses its target.

from logic; what is more, the only possible path is threaten by the paradoxes.[41] But we should not reduce *Principia* to its first parts. As one moves forward in the reading, one realizes that the existence problem is progressively supplanted by the issue raised by the existence of multiple reductions. Studying the last published parts of *Principia* does not merely allow us to fill a gap in the scholarship; it helps us to readjust our conception of the whole work.

One last word about Whitehead. I have suggested that the theory of quantity fits well the logicist agenda, once this agenda is distinguished from the arithmetization program — and that Russell could then be regarded as the coauthor of the doctrine. This being said, Whitehead's particular influence on Part VI should be acknowledged. One thing is to say that a logicist definition of numbers satisfying the application constraint can be found; another is to promote, as Whitehead did, the application constraint itself. At numerous places in Whitehead's writing, one finds the idea that the striving for generalization and abstraction in mathematics goes hand-in-hand with the wish to account for application. For example, in *Universal Algebra*, Whitehead attempted to show that Grassmannian algebra could find a natural interpretation in terms of non-Euclidean geometries, thus revealing that the more abstract and general the algebras were, the more capable of being applied they became.[42] But, closer to *Principia* VI, let me quote a passage from ([Whitehead, 1911], p. 100):

> One of the most fascinating characteristics of mathematics is the surprising way in which the ideas and results of different parts of the subject dovetail into each other. During the discussions of [...] the previous chapter we have been guided merely by the most abstract of pure mathematical considerations; and yet at the end of them we have been led back to the most fundamental of all the laws of nature, laws which have to be in the mind of every engineer as he designs an engine, and of every naval architect as he calculates the stability of a ship. It is no paradox to say that in our most theoretical moods we may be nearest to our most practical applications.

The idea that "in our most theoretical moods we may be nearest to our most practical considerations" can easily be used to justify the application constraint. Now, one does not find this emphasis on application in Russell's

[41] In fact, in *Principia*, Russell has given up the idea of deriving the infinity axiom — according to Boolos (for instance), this shows that Russell did not find the Northwest Passage.

[42] For more on this interpretation, see [Gandon, 2005] and [Desmet, 2010].

writings.[43] I am then not sure that Russell, without Whitehead's impetus, would have espoused the application constraint and would have felt the need to wander from the standard Dedekindian definition. What is truly beautiful then is that *Principia*'s logical framework was flexible and powerful enough to accommodate the difference between Whitehead's and Russell's own sources of inspiration.

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[43] For instance, that the most abstract developments always lead back to "the most fundamental of all the laws of nature" seems to be in tension with the importance Russell gave to Cantor's theory of transfinite numbers.

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